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THE APPROXIMATE DISTRIBUTIONS OF THE MEAN AND VARIANCE OF A SAMPLE OF INDEPENDENT VARIABLES

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1. Introduction. In this paper we shall study the mean and variance of a large number, n (a sample of size n) of mutually independent random variables:

$$\xi_1, \xi_2, \cdots, \xi_n, \tag{1}$$

having the same probability distribution represented by a (cumulative) distribution function $P(x)$. The rth moment, absolute moment, and semi-invariant of $P(x)$ are denoted by α_r, β_r, and γ_r respectively. It is assumed that for a certain integer $k \geq 3$, $\beta_k < \infty$ and that $\alpha_2 > 0$. Hence there is no loss of generality in assuming that

$$\alpha_1 = 0, \qquad \alpha_2 = 1. \tag{2}$$

The characteristic function corresponding to $P(x)$ is denoted by $p(t)$.

We put

$$\bar{\xi} = \frac{1}{n}\sum_{r=1}^{n} \xi_r, \qquad \eta = \frac{1}{n}\sum_{r=1}^{n} (\xi_r - \bar{\xi})^2 \tag{3}$$

$$F(x) = Pr\{\sqrt{n}\,\bar{\xi} \leq x\}, \qquad G(x) = Pr\left\{\frac{\sqrt{n}(\eta - 1)}{\sqrt{\alpha_4 - 1}} \leq x\right\}. \tag{4}$$

The definition of $G(x)$ implies that $\alpha_4 < \infty$ and $\alpha_4 - 1 > 0$. The case $\alpha_4 - 1 = 0$ provides an easy degenerated case which will be treated separately (section 4).

Cramér's theorem of asymptotic expansion[1] reads as follows:

Theorem 1. *If $P(x)$ is non-singular and if $\beta_k < \infty$ for some integer $k \geq 3$, then*

$$F(x) = \Phi(x) + \Psi(x) + R(x) \tag{5}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,dy. \tag{6}$$

$\Psi(x)$ is a certain linear combination of successive derivatives $\Phi^{(3)}(x), \cdots, \Phi^{(3(k-3))}(x)$ with each coefficient of the form $n^{-\frac{1}{2}\nu}$ times a quantity depending only on $k, \alpha_3, \cdots, \alpha_{k-1}$ $(1 \leq \nu \leq k - 3)$ and

$$|R(x)| \leq Q/n^{\frac{1}{2}(k-2)} \tag{7}$$

where Q is a constant depending only on k and $P(x)$.

[1] H. Cramér: *Random Variables and Probability Distributions* (1937), Ch. 7. This book will be referred to as (C).

In particular, putting $k = 3$ we get that $|F(x) - \Phi(x)| = O(n^{-1/2})$ provided $P(x)$ is non-singular and $\beta_3 < \infty$. If the condition of non-singularity of $P(x)$ be removed, then Liapounoff's theorem[2] furnishes the weaker result: $|F(x) - \Phi(x)| \leq A\beta_3 n^{-1/2} \log n$ where A is a numerical constant.

Very recently Berry[3] succeeded in removing the factor $\log n$ from Liapounoff's theorem under no other condition than that $\beta_3 < \infty$. We state here Berry's theorem:

THEOREM 2. *If $\beta_3 < \infty$, then*

$$(8) \qquad |F(x) - \Phi(x)| \leq \frac{A\beta_3}{\sqrt{n}}$$

where A is a numerical constant.

An essential step in the proof of these results is the selection of a weighting function $w(x)$ and the appraisal of the integral

$$(9) \qquad \int_{-\infty}^{\infty} w(u)\{F(u + x) - \Phi(u + x) - \Psi(u + x)\}\, du$$

($\Psi \equiv 0$ when $k = 3$). In his book[1] Cramér proves Theorem 1 by taking $w(u) = \frac{1}{\Gamma(\omega)}(-u)^{\omega-1}$ when $u < 0$ and $w(u) = 0$ when

$$(10) \qquad u \geq 0 \quad (0 < \omega < 1)$$

and proves Liapounoff's theorem by taking

$$(11) \qquad w(u) = \frac{1}{\sqrt{2\pi\epsilon}}\, e^{-u^2/2\epsilon^2}.$$

On the other hand, Berry uses the following weighting function in his proof of Theorem 2:

$$(12) \qquad w(u) = \frac{1 - \cos Tu}{u^2}.$$

The unfortunate selection of the function (11) accounts for the presence of the factor $\log n$ in Liapounoff's theorem.

Now Cramér's proof of Theorem 1, based on the integral (9) with $w(u)$ defined in (10), makes use of a result on that integral due to M. Riesz. A more elementary proof than this can be devised. In fact, one has only to use, with Berry, the function (12) and to adopt his elementary appraisal[4] of the integral

[2] (C), Ch. 7.

[3] A. C. BERRY: "The accuracy of the Gaussian approximation to the sum of independent variates." *Trans. Amer. Math. Soc.*, Vol. 49 (1941), pp. 122–136. This paper will be referred to as (B).

[4] Berry proves the inequality (in our notation):

$$\left|\int_{-\infty}^{\infty} \frac{1 - \cos Tx}{x^2}\{F(x + a) - \Phi(x + a)\}\, dx\right| \leq \int_0^T \frac{(T - t)\,|f(t) - e^{-\frac{1}{2}t^2}|}{t}\, dt$$

(9) in order to obtain the proof of Theorem 1. One of our purposes is therefore to give an elementary proof of Theorem 1, without reference to the above-mentioned result due to M. Riesz Section 2 is devoted to this work.

We ought to add that Cramér's theorem and Berry's theorem correspond to Theorems 1 and 2 for the case in which the random variables (1) do not follow the same distribution. The proof given in Section 2 is adaptable to these more general theorems when subjected to appropriate modifications; the assumption of a common distribution function for (1) is only made for the sake of convenience.

So much for the known results for the approximate distribution of $\bar{\xi}$. By a purely formal operational method Cornish and Fisher[5] obtain terms of successive approximation to the distribution function of any random variable X with the help of its semi-invariants. It is hardly necessary to emphasize the importance of turning Cornish and Fisher's formal result (asymptotic expansion without appraisal of the remainder) into a mathematical theorem of asymptotic expansion which gives the order of magnitude of the remainder. In this paper we achieve this for the simplest function of (1) next to $\bar{\xi}$, viz. the η in (3). We do not seek to remove the assumption of a common distribution for (1), as there will be no practical significance (e.g. in statistics) of η if the variables (1) do not have the same probability distribution. Section 3 is devoted to the proof of the following theorems:

THEOREM 3. *If $\alpha_6 < \infty$ and $\alpha_4 - 1 - \alpha_3^2 \neq 0$ (it cannot be negative), then*

$$(13) \qquad \left| G(x) - \Phi(x) \right| \leq \frac{A}{\sqrt{n}} \left(\frac{\alpha_6}{\alpha_4 - 1 - \alpha_3^2} \right)^{3/2}$$

where A is a numerical constant.

THEOREM 4. *Let $P(x)$ be non-singular and let $\alpha_{2k} < \infty$ for some integer $k > 3$. Then*

$$(14) \qquad G(x) = \Phi(x) + \chi(x) + R_1(x),$$

where $\Phi(x)$ is the function (6), $\chi(x)$ is a linear combination of the derivatives $\Phi'(x)$, $\cdots$, $\Phi^{(3(k-3))}(x)$ with each coefficient of the form $n^{-\frac{1}{2}\nu}$ times a quantity depending only on k and α_3, α_4, $\cdots$, α_{2k-2}, and

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(B), p. 128. The "appraisal" mentioned here refers to (50) which is contained in B, p 128. But Berry's appraisal of the integral in the right-hand side of the above inequality is in default. He writes

$$\frac{\epsilon}{6} \int_0^{c/\epsilon} \left(\frac{1.1}{\epsilon} - t \right) t^2 e^{-\frac{1}{2}t^2} \, dt = \frac{1.1}{6} \sqrt{\frac{\pi}{2}} - \frac{\epsilon}{3} - \frac{1}{6} \int_{c/\epsilon}^{\infty} \left\{ (1.1 - c)t^2 + c - \frac{2c}{t^2} \right\} e^{-t^2/2} \, dt$$

(B, p. 132, line 3) whilst the last integral ought to be

$$\int_{c/\epsilon}^{\infty} \{ (1.1 - c)t^2 + c - 2ct \} e^{-t^2/2} \, dt.$$

[5] E. A. Cornish and R. A. Fisher: "Moments and cumulants in the specification of distributions." (Revue de l'Institut International de Statistique (1937), pp. 1–14.)

$$(15)\qquad |R_1(x)| \leq \frac{Q_k}{n^{\frac{1}{2}(k-3)}} \quad \text{if } k = 4, 5 \text{ or } 6$$

$$(16)\qquad |R_1(x)| \leq \frac{Q'_k}{n^{k(k-1)/(2k+2)}} \quad \text{if } k \geq 7$$

where Q_k and Q'_k are constants depending only on k and $P(x)$.

It may be noticed that Theorem 3 is a "Berryian" theorem about $G(x)$, its characteristic feature being the absence of any condition on the distribution function except the two on its moments, and that Theorem 4 is a "Cramerian" theorem about $G(x)$, the characteristic feature being the assumption of non-singularity of $P(x)$ besides that $\alpha_{2k} < \infty$.

In proving these theorems we have devised a method which is applicable to getting similar results about functions other than η, such as functions commonly used in applied statistics: the higher moments about the means, the moment ratios (e.g. K. Pearson's b_1 and b_2), the covariance, the coefficient of correlation, and "Student's" t-statistic. Works on such functions are being done by my university colleagues, and the results will be published shortly.

If ξ is any of the random variables (1), then

$$0 \leq \epsilon\{a(\xi^2 - 1) + b\xi\}^2 = a^2(\alpha_4 - 1) + 2ab\alpha_3 + b^2$$

for all real (a, b). Hence $\alpha_4 - 1 - \alpha_3^2 \geq 0$, and $\alpha_4 - 1 - \alpha_3^2 = 0$ means that there is unit probability that ξ assumes exactly two values. This easily degenerated case is first eliminated in Theorem 3 by the assumption $\alpha_4 - 1 - \alpha_3^2 \neq 0$ and then considered in section 4. In Theorem 4 the condition $\alpha_4 - 1 - \alpha_3^2 \neq 0$ is implied since ξ cannot be a random variable of the nature just described owing to the non-singularity of $P(x)$.

2. Lemmas. Throughout this paper A, B, C, etc. will denote positive numerical constants; A_k, B_k (A_{km}, B_{km}), etc., will denote positive constants depending only on some integer k (integers k and m), and Q_k (Q_{km}) will denote a positive constant depending only on k (k and m) and the distribution function $P(x)$. ϑ, Θ, Θ_k, (Θ_{km}), Λ_k (Λ_{km}) will denote respectively quantities such that $|\vartheta| \leq 1$, $|\Theta| \leq A$, $|\Theta_k| \leq A_k$ ($|\Theta_{km}| \leq A_{km}$), $|\Lambda_k| \leq Q_k$ ($|\Lambda_{km}| \leq Q_{km}$). These symbols do not necessarily stand for the same quantity at each occurrence. Thus $2\vartheta = \Theta$, $k\Theta_k = \Theta_k$ etc. In particular any positive functions of $k, \alpha_3, \cdots, \alpha_k$ is a Q_k.

1.1. Cramér obtains the asymptotic expansion of the characteristic function of the distribution of $\sqrt{n}\xi$, viz. $\epsilon(e^{it\sqrt{n}\xi})$, when (1) do not have the same distribution, valid for $|t| \leq Q_k n^{1/6}$. Since we assume a common distribution for (1), so that the characteristic function is $\left\{p\left(\frac{t}{\sqrt{n}}\right)\right\}^n$, we are able to derive an asymptotic expansion valid for $|t| \leq Q_k\sqrt{n}$. The extension to $\left\{p\left(\frac{t_1}{\sqrt{n}},\right.\right.$

$\cdots, \frac{t_m}{\sqrt{n}}\Big)\Big\}^n$ presents no difficulty. This is done in the following three lemmas, of which Lemma 3 contains the final result.

LEMMA 1.

$$\log p(t) = \sum_{r=2}^{k-1} \frac{\gamma_r (it)^r}{r!} + \Theta_k \beta_k |t|^k, \quad \text{for } |t| \leq \beta_k^{1/k}. \tag{17}$$

PROOF: Since $p(t) = 1 + \sum_{r=1}^{k-1} \frac{\alpha_r (it)^r}{r!} + \frac{\vartheta \beta_k |t|^k}{k!} = 1 + q(t)$ say, we have, for $\beta_k^{1/k} |t| \leq 1$,

$$q(t) \leq \sum_{r=2}^{k} \frac{\beta_r |t|^r}{r!} \leq \sum_{r=2}^{k} \frac{(\beta_k^{1/k} |t|)^r}{r!} < \sum_{r=2}^{\infty} \frac{1}{r!} = e - 2 < \frac{3}{4}.$$

Hence

$$\log p(t) = \sum_{1 \leq j \leq [\frac{1}{2}(k-1)]} (-1)^{j+1} \frac{\{q(t)\}^j}{j} + \Theta |q(t)|^{[\frac{1}{2}(k+1)]}. \tag{18}$$

For $1 \leq j \leq [\frac{1}{2}(k-1)]$ let us expand each $(-1)^{j+1} j^{-1} \{q(t)\}^j$ to get a polynomial $q_j(t)$ of degree $k - 1$ and a remainder $r_j(t)$. In doing this we regard $q(t)$ formally as a polynomial of degree k in t. For this polynomial we have the majorating relation

$$q(t) \ll e^{\beta_k^{1/k} |t|},$$

whence

$$\frac{(-1)^j}{j} \{q(t)\}^j \ll e^{j \beta_k^{1/k} |t|},$$

which gives

$$|r_j(t)| \leq \sum_{r=k}^{\infty} \frac{j^r \beta_k^{r/k} |t|^r}{r!} \leq j^k \beta_k |t|^k e^{j \beta_k^{1/k} |t|} \leq j^k e^j \beta_k |t|^k \leq A_k \beta_k |t|^k. \tag{19}$$

Similarly,

$$|q(t)|^{[\frac{1}{2}(k+1)]} \leq A_k \beta_k |t|^k \tag{20}$$

From (18), (19), (20) we obtain

$$\log p(t) = \sum_{1 \leq j \leq [\frac{1}{2}(k-1)]} q_j(t) + \Theta_k \beta_k |t|^k. \tag{21}$$

Since the sum in (21) must equal the sum in (17), the Lemma is proved.

LEMMA 2. *Let $(\zeta_1, \zeta_2, \cdots, \zeta_m)$ be a random point with $\epsilon(\zeta_i) = 0$ and $\epsilon(|\zeta_i|^k) = \beta_{ki} < \infty$ for some integer $k \geq 3$ $(i = 1, \cdots, m)$. Let $p(t_1, \cdots, t_m)$ be the characteristic function. Then for $|t_i| \leq m^{-2+1/k} \beta_{ki}^{-1/k} \sqrt{n}$ $(i = 1, \cdots, m)$ we have*

$$n \log p\left(\frac{t_1}{\sqrt{n}}, \cdots, \frac{t_m}{\sqrt{n}}\right) = \sum_{r=2}^{k-1} \frac{i^r U_r}{r!\, n^{\frac{1}{2}(r-2)}} + \frac{\Theta_k V_k}{n^{\frac{1}{2}(k-2)}} \tag{22}$$

where U_r and V_r are the rth semi-invariant and the absolute moment respectively of $\Sigma t_i\zeta_i$

PROOF: If $|t_i| \leq m^{-2+1/k}\beta_{ki}^{-1/k}\sqrt{n}$, then $V_k^{1/k} \leq m^{(k-1)/k}(\Sigma\beta_{ki}|t_i|^k)^{1/k} \leq m^{(k-1)/k}(\Sigma\beta_{ki}^{1/k}|t_i|) \leq \sqrt{n}$. Since $p\left(\frac{t_1}{\sqrt{n}}, \cdots, \frac{t_m}{\sqrt{n}}\right)$ is the value at $t = \frac{1}{\sqrt{n}}$ of the characteristic function of $\Sigma t_i\zeta_i$, it follows from Lemma 1 that for $\sqrt{n} \geq V_k^{1/k}$ we have (22).

LEMMA 3. *Let $(\zeta_1, \cdots, \zeta_m)$ be a random point with $\epsilon(\zeta_i) = 0$, $\epsilon(\zeta_i^2) = 1$ and $\epsilon(|\zeta_i|^k) = \beta_{ki} < \infty$ for some integer $k \geq 3$. Let $\rho_{ij} = \epsilon(\zeta_i\zeta_j)(\rho_{ii} = 1; i, j = 1, \cdots, m)$ and the matrix $\|\rho_{ij}\|$ be positive definite. Let*

$$\Delta = \det.|\rho_{ij}|, \qquad \varphi(t_1, \cdots, t_m) = e^{-\frac{1}{2}\sum_{i,j=1}^{m}\rho_{ij}t_it_j}. \tag{23}$$

Let $p(t_1, \cdots, t_m)$ be the characteristic function. Then there exists a B_{km} such that for $|t_i| \leq \frac{B_{km}\Delta\sqrt{n}}{\beta_{ki}^{3/k}}$ $(i = 1, \cdots, m)$ we have

$$\begin{aligned}\left\{p\left(\frac{t_1}{\sqrt{n}}, \cdots, \frac{t_m}{\sqrt{n}}\right)\right\}^n = \varphi(t_1, \cdots, t_m)\{1 + \psi(it_1, \cdots, it_m)\} \\ + \frac{\Theta_{km}}{n^{\frac{1}{2}(k-2)}}\left\{\sum_{i=1}^{m}\beta_{ki}^{3(k-2)/k}(|t_i|^k\right. \\ \left. + |t_i|^{k+1} + \cdots + |t_i|^{3(k-2)})\right\}e^{-\Delta/4m^{m-1}\sum_{i=1}^{m}t_i^2}\end{aligned} \tag{24}$$

where $\psi(it_1, \cdots, it_m)$ is a polynomial each of whose terms has the form

$$\frac{1}{n^{\nu/2}}a_{\nu_1\cdots\nu_m}(it_1)^{\nu_1}\cdots(it_m)^{\nu_m},$$

with $1 \leq \nu \leq k-3$, $3 \leq \nu_1 + \cdots + \nu_m \leq 3(k-3)$, and $a_{\nu_1\cdots\nu_m}$ depending only on k and the moments $\epsilon(\zeta_1^{\mu_1}\cdots\zeta_m^{\mu_m})$, $3 \leq \mu_1 + \cdots + \mu_m \leq k-1$. If $k = 3$, then $\psi = 0$.

PROOF. If $|t_i| \leq m^{-2+(1/k)}\beta_{ki}^{-3/k}\Delta\sqrt{n}$, then $|t_i| \leq m^{-2+(1/k)}\beta_{ki}^{-1/k}\sqrt{n}$ since $\Delta \leq 1$ and $\beta_{ki} \geq 1$. It follows from Lemma 2 and the fact $U_2 = \Sigma\rho_{ij}t_it_j$ that

$$\begin{aligned}\left\{p\left(\frac{t_1}{\sqrt{n}}, \cdots, \frac{t_m}{\sqrt{n}}\right)\right\}^n &= \varphi(t_1, \cdots, t_m)e^s \\ &= \varphi(t_1, \cdots, t_m)\left\{1 + \sum_{j=1}^{k-3}\frac{s^j}{j!} + \frac{\vartheta|s|^{k-2}e^{|s|}}{(k-2)!}\right\}\end{aligned} \tag{25}$$

where

$$s = \frac{i^3}{\sqrt{n}}\sum_{r=0}^{k-3}\frac{i^rU_{r+3}}{(r+3)!\,n^{r/2}} + \frac{\Theta_kV_k}{n^{\frac{1}{2}(k-2)}}. \tag{26}$$

Regarding s formally as a polynomial in $n^{-\frac{1}{2}}$ let us expand each $(j!)^{-1}s^j$ $(1 \leq j \leq k-3)$ to get a polynomial s_j of degree $k-3$ in $n^{-\frac{1}{2}}$ and a remainder r_j. For the formal polynomial s we have the majorating relation

$$s \ll \frac{A_k}{\sqrt{n}} \sum_{r=0}^{k-3} \frac{V_{r+3}}{r!\,n^{r/2}} \ll \frac{A_k}{\sqrt{n}} \sum_{r=0}^{k-3} \frac{V_k^{(r+3)/k}}{r!\,n^{r/2}} \ll \frac{A_k V_k^{3/k}}{\sqrt{n}}\, e^{V_k^{1/k} n^{-\frac{1}{2}}}, \tag{27}$$

whence

$$\frac{1}{j!} s^j \ll A_k \frac{V_k^{3j/k}}{n^{j/2}}\, e^{jV_k^{1/k} n^{-\frac{1}{2}}},$$

which gives

$$|r_j| \leq \frac{A_k V_k^{3j/k}}{n^{j/2}} \sum_{\nu=k-2-j}^{\infty} \frac{j^\nu V_k^{\nu/k}}{\nu!\,n^{\nu/2}} \leq \frac{A_k V_k^{(k-2+2j)/k}}{n^{\frac{1}{2}(k-2)}}\, e^{j(V_k^{1/k}/\sqrt{n})}.$$

Since $V_k^{1/k} n^{-\frac{1}{2}} \leq 1$ as shown in the proof of Lemma 2, we have

$$|r_j| \leq \frac{A_k V_k^{(k-2+2j)/k}}{n^{\frac{1}{2}(k-2)}} \leq \frac{A_{km}(\sum_i \beta_{ki} |t_i|^k)^{(k-2+2j)/k}}{n^{\frac{1}{2}(k-2)}}$$

$$\leq \frac{A_{km}(\sum_i \beta_{ki}^{1/k} |t_i|)^{k-2+2j}}{n^{\frac{1}{2}(k-2)}} \leq \frac{A_{km} \sum \beta_{ki}^{(k-2+2j)/k} |t_i|^{k-2+2j}}{n^{\frac{1}{2}(k-2)}}.$$

Since $\beta_{ki} \geq 1$ we have $\beta_{ki}^{(k-2+2j)/k} \leq \beta_{ki}^{3(k-2)/k}$. Hence

$$|r_j| \leq \frac{A_{km} \sum_i \beta_{ki}^{3(k-2)/k} |t_i|^{k-2+2j}}{n^{\frac{1}{2}(k-2)}}. \tag{28}$$

Similarly

$$\frac{|s|^{k-2}}{(k-2)!} \leq \frac{A_{km} \sum_i \beta_{ki}^{3(k-2)/k} |t_i|^{3(k-2)}}{n^{\frac{1}{2}(k-2)}}. \tag{29}$$

From (25), (28), (29) we get

$$\left\{p\left(\frac{t_1}{\sqrt{n}}, \cdots, \frac{t_m}{\sqrt{n}}\right)\right\}^n = \varphi(t_1, \cdots, t_m)\left\{1 + \sum_{j=1}^{k-3} s_j + \sum_{j=1}^{k-3} r_j + \frac{\vartheta |s|^{k-2}}{(k-2)!} e^{|s|}\right\}$$

$$= \varphi(t_1, \cdots, t_m)\,\{1 + \psi(it_1, \cdots, it_m)\}$$

$$+ \frac{\Theta_{km}}{n^{\frac{1}{2}(k-2)}}\,\{\Sigma \beta_{ki}^{3(k-2)/k}(|t_i|^k + |t_i|^{k+1} + \cdots + |t_i|^{3(k-2)})\}\, \varphi(t_1, \cdots, t_m) e^{|s|}$$

where $\psi(it_1, \cdots, it_m)$ stands for Σs_j. The assertion about $\psi(it_1, \cdots, it_m)$ announced in the lemma can now be seen without difficulty. It remains to show that with suitable B_{km} in the lemma, we have

$$\varphi(t_1, \cdots, t_m) e^{|s|} \leq e^{-\Delta/4m m^{-1} \sum_{i=1}^{m} t_i^2}$$

i.e.

$$-\frac{1}{2}\sum_{i,j=1}^{m}\rho_{ij}t_i t_j + |s| \leq -\frac{\Delta}{4m^{m-1}}\sum_{i=1}^{m} t_i^2. \tag{30}$$

From (27) we have

$$\begin{aligned} |s| &\leq \frac{A_k}{\sqrt{n}}\Gamma_k^{3/k} \leq \frac{A_{km}}{\sqrt{n}}\left(\sum_i \beta_{ki}|t_i|^k\right)^{3/k} \\ &\leq \frac{A_{km}}{\sqrt{n}}\left(\sum_i \beta_{ki}^{1/k}|t_i|\right)^3 \leq \frac{A_{km}}{\sqrt{n}}\sum_i \beta_{ki}^{3/k}|t_i|^3. \end{aligned} \tag{31}$$

If we choose $B_{km} \leq (4m^{m-1}A_{km})^{-1}$ (and $B_{km} \leq m^{-2+(1/k)}$ in order that the earlier results may not be affected), the A_{km} here coinciding with the last written A_{km} in (31), we have, for $|t_i| \leq B_{km}\beta_{ki}^{-3/k}\Delta\sqrt{n}$,

$$|s| \leq \frac{\Delta}{4m^{m-1}}\sum_{i=1}^{m} t_i^2. \tag{32}$$

On the other hand, if $\lambda_1, \lambda_2, \cdots, \lambda_m$ are the latent roots of $\| \rho_{ij} \|$ then each $\lambda_i \leq m$ since their sum is m. Letting λ_1 be the smallest one we have

$$\frac{1}{2}\sum_{i,j}\rho_{ij}t_i t_j \geq \frac{1}{2}\lambda_1\sum t_i^2 = \frac{\lambda_1\lambda_2\cdots\lambda_m}{2\lambda_2\cdots\lambda_m}\sum t_i^2 \geq \frac{\Delta}{2m^{m-1}}\sum t_i^2. \tag{33}$$

(32) and (33) imply (30). Hence the lemma is proved.

Let us write down the particular cases $m = 1$ and $m = 2$ of (24):

$$\begin{aligned} \left\{p\left(\frac{t}{\sqrt{n}}\right)\right\}^n &= e^{-\frac{1}{2}t^2}(1 + \psi(it)) \\ &+ \frac{\Theta_k}{n^{\frac{1}{2}(k-2)}}\beta_k^{3(k-2)/k}\{|t|^k + |t|^{k+1} + \cdots + |t|^{3(k-2)}\}e^{-t^2/4}, \left(|t| \leq \frac{A_k\sqrt{n}}{\beta_k^{3/k}}\right) \end{aligned} \tag{34}$$

$$\begin{aligned} \left\{p\left(\frac{t_1}{\sqrt{n}}, \frac{t_2}{\sqrt{n}}\right)\right\}^n &= e^{-\frac{1}{2}(t_1^2+t_2^2+2\rho t_1 t_2)}\{1 + \psi(it_1, it_2)\} \\ &+ \frac{\Theta_k}{n^{\frac{1}{2}(k-2)}}\left\{\sum_{i=1}^{2}\beta_{ki}^{3(k-2)/k}(|t_i|^k + |t_i|^{k+1} + \cdots + |t_i|^{3(k-2)})\right\}e^{-(1-\rho^2)(t_1^2+t_2^2)/8} \\ &\left(|t_i| \leq \frac{A_k(1-\rho^2)\sqrt{n}}{\beta_{ki}^{3/k}}, \quad \rho = \epsilon(\zeta_1\zeta_2)\right). \end{aligned} \tag{35}$$

More specially let us rewrite (34) and (35) with $k = 3$:

$$\left\{p\left(\frac{t}{\sqrt{n}}\right)\right\}^n = e^{-\frac{1}{2}t^2} + \frac{\Theta}{\sqrt{n}}\beta_3|t|^3e^{-\frac{1}{4}t^2}, \qquad \left(|t| \leq \frac{A\sqrt{n}}{\beta_3}\right); \tag{36}$$

$$\begin{aligned} \left\{p\left(\frac{t_1}{\sqrt{n}}, \frac{t_2}{\sqrt{n}}\right)\right\}^n &= e^{-\frac{1}{2}(t_1^2+t_2^2+2\rho t_1 t_2)} \\ &+ \frac{\Theta}{\sqrt{n}}(\beta_{31}|t_1|^3 + \beta_{32}|t_2|^3)e^{-(1-\rho^2)(t_1^2+t_2^2)/8}, \qquad \left(|t_i| \leq \frac{A(1-\rho^2)\sqrt{n}}{\beta_{3i}}\right). \end{aligned} \tag{37}$$

In this paper only these last four formulae are needed; they are used in the proofs of Theorems 2, 1, 3, 4 respectively. Cases of $m > 2$ of (24) will be needed for the works on other functions alluded to in the introduction.

1.2. In the following group of lemmas, which culminate in Lemma 7, one finds a generalization of the Riemann-Lebesgue theorem, viz. Lemma 6.

LEMMA 4. *Let $f(x)$ be a polynomial of degree $m > 0$, with real coefficients:*

$$f(x) = \sum_{i=0}^{m} a_i x^{m-i} \qquad (a_0 \neq 0) \tag{38}$$

Then

$$\left| \int_0^1 e^{if(x)}\, dx \right| \leq \frac{A_m}{|a_0|^{1/m}}. \tag{38}$$

PROOF: It is sufficient to prove the inequality for $\int_0^1 \cos f(x)\, dx$. Divide the interval into A_m sub-intervals in each of whose interior none of the derivatives $f^{(i)}(x)$ $(i = 1, \cdots, m)$ vanishes. It is sufficient to consider one of these sub-intervals, say (a, b). Consequently each of the polynomials $f^{(i)}(x)$ are monotonic in (a, b). Let

$$I = \int_a^b \cos f(x)\, dx. \tag{39}$$

Suppose first that $f'(x)$ is positive and increasing for $a < x \leq b$. Then

$$|I| \leq \epsilon + \left| \int_{a+\epsilon}^{b} \frac{f'(x) \cos f(x)\, dx}{f'(x)} \right|$$
$$= \epsilon + \frac{1}{f'(a+\epsilon)} \left| \int_{a+\epsilon}^{b_1} f'(x) \cos f(x)\, dx \right|, \quad (a + \epsilon \leq b_1 \leq b),$$

by the second mean-value theorem. Hence

$$|I| \leq \epsilon + \frac{2}{f'(a+\epsilon)}. \tag{40}$$

Now $0 < f'(a + \frac{1}{2}\epsilon) = f'(a + \epsilon) - \epsilon f''(a + \theta\epsilon)/2$, $\frac{1}{2} \leq \theta \leq 1$. Hence $f'(a + \epsilon) > \frac{1}{2}\epsilon f''(a + \theta\epsilon)$. Since $f''(x)$ is monotonic, we have either $f'(a + \epsilon) > \frac{1}{2}\epsilon f''(a + \epsilon)$ or $f'(a + \epsilon) > \frac{1}{2}\epsilon f''(a + \frac{1}{2}\epsilon)$. In other words, there exists a constant C_2, independent of a or ϵ, such that $\frac{1}{2} \leq C_2 \leq 1$ and $f'(a + \epsilon) > \frac{1}{2}\epsilon f(a + C_2\epsilon)$.

If $f'''(x) \geq 0$, we have, as before $f''(a + C_2\epsilon) > \frac{1}{2}C_2\epsilon f'''(a + C_3\epsilon)$, where C_3 is independent of a or ϵ and $\frac{1}{4} \leq C_3 \leq 1$. If $f'''(x) < 0$, then, since $0 < f''(a + 2C_2\epsilon) = f''(a + C_2\epsilon) + C_2\epsilon f'''(a + \theta_1 C_2\epsilon)$, $\frac{1}{2} \leq \theta_1 \leq 1$, we have $f''(a + C_2\epsilon) > -C_2\epsilon f'''(a + 2\theta_1 C_2\epsilon)$. As $f'''(x)$ is monotonic, either $f''(a + C_2\epsilon) > -C_2\epsilon f'''(a + C_2\epsilon)$ or $f''(a + C_2\epsilon) > -C_2\epsilon f'''(a + 2C_2\epsilon)$. In all cases we obtain $f''(a + C_2\epsilon) > B_3\epsilon\, |f'''(a + C_3\epsilon)|$, where B_3 and C_3 are independent of a or ϵ, and $\frac{1}{4} \leq C_3 \leq 2$. Hence $f'(a + \epsilon) > \frac{1}{2}B_3\epsilon^2\, |f'''(a + C_3\epsilon)|$. Arguing with $\pm f'''(a + C_3\epsilon)$ as we did with $f''(a + C_2\epsilon)$, and so on until we come to $f^{(m)}$,

we obtain $f'(a+\epsilon) > B_m\epsilon^{m-1}|a_0|$, $f'(x) \ge f'(a+C_m\epsilon) \ge B_m\epsilon^{m-1}|a_0|$. Substituting in (40) and putting $\epsilon = |a_0|^{-1/m}$ we obtain $|I| \le A_m|a_0|^{-1/m}$. The proof presupposes that $C_m\epsilon \le b - a$. If the reverse inequality is true, then $|I| \le b - a < C_m|a_0|^{-1/m}$. Hence the lemma is true for $f'(x)$ positive and increasing in (a, b).

If $f'(x)$ is positive and decreasing in (a, b), then $I = \int_0^{b-a} \cos(-f(b-y))\,dy$, $-f(b-y)$ being a polynomial with the leading coefficient $\pm a_0$ and the first derivative $f'(b-y)$, which is positive and increasing. This case reduces therefore to the preceding one. Finally, if $f'(x)$ is negative, we have only to notice that $I = \int_a^b \cos(-f(x))\,dx$. Hence the lemma is proved.

Lemma 5. *Let $f(x)$ be the polynomial (38a), and let $a_r \neq 0$ for some r, $0 \le r \le m$. Then*

$$\left|\int_0^1 e^{if(x)}\,dx\right| \le \frac{A_m}{|a_r|^{B_m}}. \tag{41}$$

Proof: We may assume that $|a_r| \ge 1$, (41) being trivial if $|a_r| < 1$. If $r = 0$ this reduces to Lemma 4. Suppose that the lemma is true for $a_0, a_1, \cdots, a_{r-1}$. Let $f_1(x) = a_0x^m + \cdots + a_{r-1}x^{m-r+1}$, $f_2(x) = f(x) - f_1(x)$ and divide $(0, 1)$ into A_m sub-intervals in each of which $f_1(x)$ is monotonic. It is sufficient to consider one of these sub-intervals, say, (a, b). We have

$$I = \int_a^b \cos\{f_1(x) + f_2(x)\}\,dx$$

$$= \int_a^b \cos f_1(x)\cos f_2(x)\,dx - \int_a^b \sin f_1(x)\sin f_2(x)\,dx.$$

We have only to consider the integral of cosines, say J. Divide (a, b) into subintervals in each of whose interior $\cos f_1(x)$ is monotonic and does not vanish. The number of such intervals does not exceed $(\frac{1}{2}\pi)^{-1}|f_1(b) - f_1(a)| \le (\frac{1}{2}\pi)^{-1}(|f_1(b)| + |f_1(a)|) < 2(|a_0| + \cdots + |a_{r-1}|)$. Then, by the second mean-value theorem,

$$|J| \le 2(|a_0| + \cdots + |a_{r-1}|)\left|\int_a^{b_1} \cos f_2(x)\,dx\right| \qquad (a \le b_1 \le b).$$

Hence, applying Lemma 4 to $f_2(x)$, we get

$$|I| \le \frac{A_m(|a_0| + \cdots + |a_{r-1}|)}{|a_r|^{1/(m-r)}} \le \frac{A_m(|a_0| + \cdots + |a_{r-1}|)}{|a_r|^{1/m}}. \tag{42}$$

On the hypothesis of induction we have $|I| \le A_m|a_i|^{-B_m}$ $(i = 0, \cdots, r-1)$. If $|a_i| \ge |a_r|^{1/2m}$ for some $i < r$, then $|I| \le A_m|a_r|^{-B_m/2m}$; if $|a_i| < |a_r|^{1/2m}$, then by (42), $|I| \le A_m|a_r|^{-1/2m}$. The proof is therefore complete.

LEMMA 6. *Let $f(x)$ be the polynomial* (38a) *and $g(x)$ be summable over $(-\infty, \infty)$. Then for every r we have*

$$\lim_{|a_r|\to\infty} \int_{-\infty}^{\infty} e^{if(x)} g(x)\, dx = 0, \quad \textit{uniformly in } a_i (i \neq r). \tag{43}$$

PROOF: By Lemma 5 We have

$$\lim_{|a_r|\to\infty} \int_0^1 e^{if(x)}\, dx = 0, \quad \text{uniformly in } a_i (i \neq r).$$

Hence

$$\lim_{|a_r|\to\infty} \int_a^b e^{if(x)}\, dx = 0, \quad \text{uniformly in } a_i (i \neq r) \tag{44}$$

for if $a \neq 0$ and $b \neq 0$, then (a, b) is the sum or the difference of two intervals of the form $(0, c)$ or $(c, 0)$, and for the latter intervals the transformation $x = \pm cy$ reduces the interval of integration to $(0, 1)$.

Let G be any open set of finite measure. Then G is the sum of a sequence $\{I_\nu\}$ of non-overlapping intervals. Since $\Sigma mI_\nu = mG < \infty$, we have

$$\sum_{\nu \geq n} mI_\nu < \epsilon, \qquad n \geq N.$$

Hence

$$\left| \int_G e^{if(x)}\, dx \right| < \epsilon + \sum_{\nu=1}^{N} \left| \int_{I_\nu} e^{if(x)}\, dx \right|$$

which, together with (44), implies

$$\lim_{|a_r|\to\infty} \int_G e^{if(x)}\, dx = 0 \quad \text{uniformly in } a_i (i \neq r). \tag{45}$$

Let S be any set of finite measure. Then there is an open set G such that $G \supset S$ and $m(G - S) < \epsilon$. Hence

$$\left| \int_S e^{if(x)}\, dx \right| < \epsilon + \left| \int_G e^{if(x)}\, dx \right|.$$

Hence, by (45),

$$\lim_{|a_r|\to\infty} \int_S e^{if(x)}\, dx = 0 \quad \text{uniformly in } a_i (i \neq r). \tag{46}$$

Now let $h(x)$ be any positive "simple" summable function, i.e. $h(x) = a_\nu > 0$ for $x \epsilon S$ $(\nu = 1, 2, \cdots, n)$ and $h(x) = 0$ otherwise. Since $h(x)$ is summable, each S_ν must be of finite measure. Hence

$$\left| \int_{-\infty}^{\infty} e^{if(x)} h(x)\, dx \right| \leq \sum_{\nu=1}^{n} a_\nu \left| \int_{S_\nu} e^{if(x)}\, dx \right|$$

which, together with (46), implies

$$\lim_{|a_r|\to\infty} \int_{-\infty}^{\infty} e^{if(x)} h(x)\, dx = 0 \quad \text{uniformly in } a_i (i \neq r).$$

Finally, let $g(x)$ be any summable function ≥ 0. Then by a well-known theorem[6] we have $g(x) = \lim h_n(x)$, where $\{h_n(x)\}$ is an ascending sequence of positive summable simple functions. Hence

$$\left|\int_{-\infty}^{\infty} e^{itf(x)} g(x)\, dx\right| \leq \left|\int_{-\infty}^{\infty} e^{itf(x)} h_n(x)\, dx\right| + \int_{-\infty}^{\infty} (g(x) - h_n(x))\, dx.$$

By monotonic convergence the last integral tends to 0 as $n \to \infty$. Hence

$$\left|\int_{-\infty}^{\infty} e^{itf(x)} g(x)\, dx\right| \leq \epsilon + \left|\int_{-\infty}^{\infty} e^{itf(x)} h_n(x)\, dx\right|,$$

which implies (43). If $g(x)$ is any summable function, we have only to consider the customary expression of $g(x)$ as the difference of two non-negative functions. This completes the proof.

LEMMA 7. *Let $P(x)$ be a non-singular distribution function of a random variable X, and let*

$$p(t_1, t_2, \cdots, t_m) = \int_{-\infty}^{\infty} e^{i \sum_{r=1}^{m} t_r x^r}\, dP. \tag{47}$$

Then for every r and every positive constant c we have

$$\underset{|t_r| \geq c}{\text{l.u.b.}} |p(t_1, \cdots, t_m)| < 1. \tag{48}$$

PROOF: We have $P(x) = a_1 P_1(x) + a_2 P_2(x)$, where $P_1(x)$ is absolutely continuous, P_2 is singular, $a_1 > 0$, $a_1 + a_2 = 1$. Hence

$$|p(t_1, t_2, \cdots, t_m)| \leq a_1 \left|\int_{-\infty}^{\infty} e^{i \sum_{r=1}^{m} t_r x^r} P_1'(x)\, dx\right| + a_2.$$

By Lemma 6 we may find $C > 0$ such that

$$|p(t_1, t_2, \cdots, t_m)| \leq \tfrac{1}{2}a_1 + a_2 < 1, \quad \text{if any} \quad |t_i| > C.$$

Suppose that

$$\underset{|t_r| \geq c}{\text{l.u.b.}}\, p(t_1, \cdots, t_m) = 1,$$

then $c < C$ and we must have

$$\underset{c \leq |t_r| \leq C, |t_i| \leq C (i \neq r)}{\text{l.u.b.}} |p(t_1, \cdots, t_m)| = 1. \tag{49}$$

Since $p(t_1, \cdots, t_m)$ is a continuous function, it must attain its least upper bound in any bounded closed set. It follows that there is a point $(t_1^0, \cdots, t_m^0)$ such that[7] $t_r^0 \neq 0$ $(|t_r^0| \geq c)$ and $p(t_1^0, \cdots, t_m^0) = 1$. But this implies that the distribution of $\Sigma t_i^0 X^i$ is discrete, i.e. that the distribution of X itself is discrete,

[6] H. Kestelman: *Modern Theories of Integration* (1937), p. 108.

[7] Cf. (C), p. 26.

which contradicts the non-singularity of $P(x)$. Hence (49) is false and (48) is true.

1.3. In his cited work Berry[8] shows that if $F(x)$ is any distribution function and if $\Phi(x)$ is the function (6), then there is a constant a such that

$$\left| \int_{-\infty}^{\infty} \frac{1 - \cos Tx}{x^2} \{F(x + a) - \Phi(x + a)\}\, dx \right| \geq \sqrt{\frac{2}{\pi}}\, T\delta \left\{ 3 \int_0^{T\delta} \frac{1 - \cos x}{x^2}\, dx - \pi \right\} \tag{50}$$

where $\delta = \sqrt{\frac{\pi}{2}}\ \mathrm{l.u.b.}\ | F(x) - \Phi(x) |$. This is easily extended to the following lemma, which needs no further proof.

LEMMA 8. *Let $F(x)$ be a distribution function and $F_1(x)$ be a function having the following properties:* (i) *$F_1(x)$ is bounded for all x,* (ii) *$F_1(x) \to 1$ as $x \to \infty$, $F_1(x) \to 0$ as $x \to -\infty$,* (iii) *$F_1(x)$ has a bounded derivative, $| F_1'(x) | \leq M$. Let*

$$\delta = \frac{1}{2M}\ \mathrm{l.u.b.}\, | F(x) - F_1(x) |.$$

Then there exists a constant a such that

$$\left| \int_{-\infty}^{\infty} \frac{1 - \cos Tx}{x^2} \{F(x + a) - F_1(x + a)\}\, dx \right| \geq 2MT\delta \left\{ 3 \int_0^{T\delta} \frac{1 - \cos x}{x^2}\, dx - \pi \right\}. \tag{51}$$

1.4. In section 3 we define, for given ϵ, k, λ and z, a function

$$G(x, y) = e^{-\epsilon y^{2k}} \quad \text{if} \quad z < x \leq z + \lambda y^2, \quad G(x, y) = 0 \quad \text{otherwise.} \tag{52}$$

The introduction of $G(x, y)$ and the appraisal of its Fourier transform constitute the essence of our method of solving the problem of the asymptotic expansion of the distribution function $G(x)$. The solution of the same problem about other functions of (1) alluded to in section 3 is based on the introduction of functions playing the role of $G(x, y)$. We now prove the following lemma:

LEMMA 9. *Let $G(x, y)$ be defined by* (52) *and let*

$$g(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-it_1x - it_2y} G(x, y)\, dx\, dy. \tag{53}$$

Then

(i) $$| g(t_1, t_2) | \leq \frac{\lambda A_k}{\epsilon^{3/2k}}$$

(ii) $$| g(t_1, t_2) | \leq \frac{A}{| t_2 |^3} \left(\lambda + \frac{\lambda^2 | t_1 |}{\epsilon^{1/3}} + \frac{\lambda^3 | t_1 |^2}{\epsilon^{2/3}} \right) \quad \text{if } k = 3,$$

(iii) $$| g(t_1, t_2) | \leq \frac{A_k}{| t_2 |^2} \left(\frac{\lambda}{\epsilon^{1/2k}} + \frac{\lambda^2 | t_1 |}{\epsilon^{3/2k}} \right).$$

[8] (B), p. 128.

PROOF:

(i) $$|g(t_1, t_2)| \le \int_{R_2} G(x, y)\,dx\,dy = \lambda \int_{-\infty}^{\infty} y^2 e^{-\epsilon y^{2k}}\,dy = \frac{A_k\lambda}{\epsilon^{3/2k}}$$

(ii) Putting $k = 3$ we have

$$g(t_1, t_2) = \frac{e^{-it_1x}}{it_1} \int_{-\infty}^{\infty} e^{-\epsilon y^6 - it_2y}(1 - e^{-it_1\lambda y^2})\,dy,$$

$$|g(t_1, t_2)| \le \frac{1}{|t_1|\,|t_2|^3} \left| \int_{-\infty}^{\infty} u(y)v'''(y)\,dy \right|,$$

where $u(y) = e^{-\epsilon y^6}(1 - e^{-it_1\lambda y^2})$, $v(y) = e^{-it_2y}$. On integrating by parts we obtain

(54) $$|g(t_1, t_2)| \le \frac{1}{|t_1|\,|t_2|^3} \left| \int_{-\infty}^{\infty} v(y)u'''(y)\,dy \right| \le \frac{1}{|t_1|\,|t_2|^3} \int_{-\infty}^{\infty} |u'''(y)|\,dy.$$

Elementary calculation establishes that

$$\frac{|u'''(y)|}{|t_1|} \le e^{-\epsilon y^6}(216\lambda\epsilon^3|y|^{17} + 756\lambda\epsilon^2|y|^{11} + 336\lambda\epsilon|y|^5 + 8\lambda^3|t_1|^2|y|^3 + 12\lambda^2|t_1|\,|y|).$$

Substituting in (54) and making the transformation $y = \epsilon^{-1/6}x$ we get the result.

(iii) We have

$$|g(t_1, t_2)| \le \frac{1}{|t_1|} \left| \int_{-\infty}^{\infty} e^{-\epsilon y^{2k} - it_2y}(1 - e^{-it_1\lambda y^2})\,dy \right|.$$

Integrating by parts twice we obtain

$$|g(t_1, t_2)| \le \frac{1}{|t_1|\,|t_2|^2} \int_{-\infty}^{\infty} \left| \frac{d^2}{dy^2} \{e^{-\epsilon y^{2k}}(1 - e^{-it_1\lambda y^2})\} \right| dy.$$

By elementary calculations we get

$$|g(t_1, t_2)| \le \frac{1}{|t_2|^2} \int_{-\infty}^{\infty} (4k^2\lambda\epsilon y^{4k} + 2k(k+3)\lambda\epsilon y^{2k} + 4\lambda^2|t_1|\,y^2 + 2\lambda)e^{-\epsilon y^{2k}}\,dy$$

which, on the transformation $y = \epsilon^{-1/2k}x$, gives the result.

1.5. We prove a few additional lemmas used in the proof of Theorems 3 and 4.

LEMMA[9] 10. *Let $u(x_1, \cdots, x_m) \ge 0$ be summable in the m-dimensional space and let*

(55) $$v(t_1, \cdots, t_m) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-it_1x_1 - \cdots - it_mx_m} u(x_1, \cdots, x_m)\,dx_1 \cdots dx_m.$$

[9] Although the author believes that this lemma is almost classical, a proof is given owing to lack of reference.

If $v(t_1, \cdots, t_m)$ is summable in the m-dimensional space, then

$$u(x_1, \cdots, x_m) = \frac{1}{(2\pi)^m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{it_1x_1 + \cdots + it_mx_m} v(t_1, \cdots t_m)\, dt_1 \cdots dt_m. \tag{56}$$

PROOF: Except for a constant factor the function $u(x_1, \cdots, x_m)$ may be regarded as a probability density function. Hence by the well-known inversion formula of (55),

$$\begin{aligned} \int \cdot \cdot \int_{a_i \le x_i \le b_i \ (i=1,\cdots,m)} u(x_1, \cdots, x_m)\, dx_1 \cdots dx_m \\ = \frac{1}{(2\pi)^m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\prod_{j=1}^{m} \frac{e^{it_jb_j} - e^{it_ja_j}}{it_j} \right) v(t_1, \cdots, t_m)\, dt_1 \cdots dt_m . \end{aligned} \tag{57}$$

Now $u(x_1, \cdots, x_m)$ is almost everywhere the symmetric derivative of the interval function in the left-hand side of (57):

$$u(x_1, \cdots, x_m) = \lim_{\epsilon \to 0} \frac{1}{(2\epsilon)^m} \int \cdots \int_{x_i - \epsilon \le y_i \le x_i + \epsilon \ (i=1,2,\cdots,m)} u(y_1, \cdots, y_m)\, dy_1 \cdots dy_m .$$

Hence

$$\begin{aligned} u(x_1, \cdots, x_m) = \frac{1}{(2\pi)^m} \lim_{\epsilon \to 0} \frac{1}{(2\epsilon)^m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \\ \cdot \left(\prod_{j=1}^{m} \frac{e^{it_j\epsilon} - e^{-it_j\epsilon}}{it_j} \right) e^{it_1x_1 + \cdots + it_mx_m} v(t_1, \cdots, t_m)\, dt_1 \cdots dt_m . \end{aligned} \tag{58}$$

Owing to dominated convergence the order of the limit sign and the integration sign in (58) may be inverted: Hence (56) is true.

LEMMA 11. *We have*

$$\int_{-\infty}^{\infty} e^{-itu} \frac{1 - \cos Tu}{u^2}\, du = \begin{cases} \pi(T - |t|) & \textit{if } |t| \le T, \\ 0 & \textit{if } |t| > T. \end{cases} \tag{59}$$

PROOF: The Fourier transform of the function in the right-hand side of (59) is

$$\pi \int_{-T}^{T} e^{itu}(T - |t|)\, dt = \frac{2\pi}{u^2}(1 - \cos Tu).$$

Hence (59) follows from (56).

LEMMA 12.

$$| \epsilon(\xi_1 + \cdots + \xi_n)^k | \le A_k n^{k/2} \beta_k \tag{60}$$

PROOF. As (60) is true for $k = 1$, let us assume, for induction, that it is true for $1, 2, \cdots, k$. Then, by symmetry,

$$\epsilon(\xi_1 + \cdots + \xi_n)^{k+1} = n\epsilon\{\xi_1(\xi_1 + \cdots + \xi_n)^k\} = n \sum_{r=0}^{k} \binom{k}{r} \epsilon(\xi_1^{r+1} \zeta^{k-r})$$

where $U = \xi_2 + \cdots + \xi_n$. Since $\epsilon(\xi_1) = 0$, we have

$$\epsilon(\xi_1 + \cdots + \xi_n)^{k+1} = n \sum_{r=1}^{k} \binom{k}{r} \epsilon(\xi_1^{r+1} U^{k-r}).$$

On the hypotheses of induction we have $|\epsilon(U^{k-r})| \leq A_k(n-1)^{\frac{1}{2}(k-r)}\beta_{k-r} < A_k n^{\frac{1}{2}(k-r)}\beta_{k-r}$. Hence

$$\left|\epsilon(\xi_1 + \cdots + \xi_n)^{k+1}\right| \leq k! A_k n^{\frac{1}{2}(k+1)} \Sigma \beta_{r+1}\beta_{k-r} \leq A_{k+1} n^{\frac{1}{2}(k+1)} \beta_{k+1}.$$

Therefore the induction is complete.

3. Elementary Proof of Theorem 1. **3.1** We have defined

$$F(x) = Pr\{\sqrt{n}\xi \leq x\}, \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy \tag{61}$$

with the characteristic functions

$$f(t) = \left\{p\left(\frac{t}{\sqrt{n}}\right)\right\}^n, \qquad \varphi(t) = e^{-\frac{1}{2}t^2}. \tag{62}$$

Following Berry[10] we use the equation

$$\int_{-\infty}^{\infty} \{F(x) - \Phi(x)\} e^{itx}\, dx = \frac{f(t) - \varphi(t)}{-it}. \tag{63}$$

Let $\psi(it)$ be the polynomial in (34), and let us define $\Psi(x)$ as the function obtained from $\psi(it)$ through the replacement of each power $(it)^r$ by $(-1)^r\Phi^{(r)}(x)$. Integration by parts shows $(-1)^{r-1} \int_{-\infty}^{\infty} e^{itx}\Phi^{(r)}(x)\, dx = (it)^{r-1}\varphi(t)$, whence

$$\int_{-\infty}^{\infty} \Psi(x) e^{itx}\, dx = \frac{\psi(it)\varphi(t)}{-it}. \tag{64}$$

From (63) and (64) we obtain

$$\int_{-\infty}^{\infty} \{F(x) - \Phi(x) - \Psi(x)\} e^{itx}\, dx = \frac{f(t) - \varphi(t)\{1 + \psi(it)\}}{-it}. \tag{65}$$

The function $\Psi(x)$ defined here is precisely the $\Psi(x)$ appearing in (5) under Theorem 1. Our task is to prove that

$$\left| F(x) - \Phi(x) - \Psi(x) \right| \leq \frac{Q_k}{n^{(k-2)/2}}. \tag{66}$$

Following Berry[11] we replace x by $x + a$ in (65), getting

$$\begin{aligned} &\int_{-\infty}^{\infty} \{F(x+a) - \Phi(x+a) - \Psi(x+a)\} e^{itx}\, dx \\ &\qquad = \frac{e^{-ita}[f(t) - \varphi(t)\{1 + \psi(it)\}]}{-it} \end{aligned} \tag{67}$$

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[10] (B), p. 127, Equation (23).

[11] (B), p. 127.

multiply both sides of (67) by $T - |t|$ and integrate with respect to t in $(-T, T)$:

$$2\int_{-\infty}^{\infty} \frac{1 - \cos Tx}{x^2} \{F(x + a) - \Phi(x + a) - \Psi(x + a)\}\, dx$$
$$= \int_{-T}^{T} \frac{(T - |t|)e^{-ita}[f(t) - \varphi(t)\{1 + \psi(it)\}]}{-it}\, dt$$

the reversion of order of integration involved is obviously justifiable. Hence

$$(68)\quad \left|\int_{-\infty}^{\infty} \frac{1 - \cos Tx}{x^2} \{F(x + a) - \Phi(x + a) - \Psi(x + a)\}\, dx\right|$$
$$\leq T \int_0^T \frac{|f(t) - \varphi(t)\{1 + \psi(it)\}|}{t}\, dt.$$

2.2. When in particular $k = 3$, (68) becomes

$$(69)\quad \left|\int_{-\infty}^{\infty} \frac{1 - \cos Tx}{x^2} \{F(x + a) - \Phi(x + a)\}\, dx\right| \leq T\int_0^T \frac{|f(t) - \varphi(t)|}{t}\, dt.$$

If we choose a to be the a in (50), the left-hand side of (69) is not less than

$$\sqrt{\frac{2}{\pi}}\, T\delta \left\{3\int_0^{T\delta} \frac{1 - \cos x}{x^2}\, dx - \pi\right\}, \qquad \delta = \sqrt{\frac{\pi}{2}}\, \text{l.u.b.}\, |F(x) - \Phi(x)|.$$

On the other hand, taking $T = \dfrac{A\sqrt{n}}{\beta_3}$ as in (36) the right-hand side of (69) is not greater than

$$A\int_0^{\infty} t^2 e^{-\frac{1}{4}t^2}\, dt = A.$$

Hence

$$(70)\qquad T\delta \left\{3\int_0^{T\delta} \frac{1 - \cos x}{x^2}\, dx - \pi\right\} \leq A.$$

Now the left-hand side of (70), as a function of $T\delta$, is positive and increasing for sufficiently large $T\delta$, and becomes infinite as $T\delta \to \infty$. Hence (70) implies that $T\delta \leq A$, i.e.

$$\text{l.u.b.}\, |F(x) - \Phi(x)| \leq \frac{A}{T} = \frac{A\beta_3}{\sqrt{n}},$$

giving Theorem 2.

2.3. Coming back to the general case, we see that the function $\Phi(x) + \Psi(x)$ has a bounded derivative: $|\Phi'(x) + \Psi'(x)| \leq Q_k$, and also has all the properties of the function $F_1(x)$ in Lemma 8. On choosing a in (69) to be the a in (51) we obtain

$$Q_k T\delta\left\{3\int_0^{T\delta}\frac{1-\cos x}{x^2}\,dx-\pi\right\}\le T\int_0^{T}\frac{|f(t)-\varphi(t)\{1+\psi(it)\}|}{t}\,dt, \tag{71}$$

where

$$\delta = Q_k \text{ l.u.b. } |F(x)-\Phi(x)-\Psi(x)|.$$

Let us take $T=(A_k\beta_k^{-3/k}\sqrt{n})^{k-2}$ with A_k in accordance with (34). Then

$$\begin{aligned} T\int_0^{T}&\frac{|f(t)-\varphi(t)\{1+\psi(it)\}|}{t}\,dt \\ &= Q_k n^{\frac12(k-2)}\int_0^{T^{1/(k-2)}} + Q_k n^{\frac12(k-2)}\int_{Q_k\sqrt{n}}^{T} = J_1+J_2 \quad \text{say.}\end{aligned} \tag{72}$$

By (34) we have

$$J_1\le Q_k\int_0^\infty (t^{k-1}+\cdots+t^{3k-7})e^{-\frac12 t^2}\,dt = Q_k. \tag{73}$$

Also,

$$J_2\le Q_k n^{\frac12(k-2)}\int_{Q_k\sqrt{n}}^{T}\frac{|p(t/\sqrt{n})|^n}{t}\,dt + Q_k n^{\frac12(k-2)}\int_{Q_k\sqrt{n}}^{T}\frac{|\varphi(t)\{1+\psi(it)\}|}{t}\,dt. \tag{74}$$

The second term in the right-hand side of (74) is evidently $\le Q_k$. The first term does not exceed

$$Q_k n^{\frac12(k-3)}\,T \underset{t\ge Q_k}{\text{l.u.b.}}\,|p(t)|^n. \tag{75}$$

At this step we make use of the non-singularity of $P(x)$ and apply Lemma 7 for $m=1$. We have

$$\underset{t\ge Q_k}{\text{l.u.b.}}\,|p(t)| = e^{-Q_k}.$$

Hence (75) does not exceed $Q_k n^{\frac12(2k-5)}e^{-Q_k n}\le Q_k$. We have therefore

$$T\delta\left\{3\int_0^{T\delta}\frac{1-\cos x}{x^2}\,dx-\pi\right\}\le Q_k, \qquad T=Q_k n^{\frac12(k-2)}. \tag{76}$$

Arguing with (76) as we did with (70) we conclude that

$$\text{l.u.b.}\,|F(x)-\Phi(x)-\Psi(x)|\le\frac{Q_k}{T}=\frac{Q_k}{n^{\frac12(k-2)}}.$$

(72) is valid for $T\ge 1$. If $T<1$, we have only to suppress the term J_2. Hence Theorem 1 is proved.

4. Proof of Theorem 3 and Theorem 4. 3.1. In connection with the random variables (1), we assume that $\beta_{2k}<\infty$ for some integer $k\ge 3$ and define

$$\eta=\frac1n\sum_{r=1}^{n}(\xi_r-\bar\xi)^2, \qquad G(z)=Pr\left\{\frac{\sqrt{n}(\eta-1)}{\sqrt{\alpha_4-1}}\le z\right\}. \tag{77}$$

Now,

$$\eta = \frac{1}{n} \sum \xi_r^2 - \bar{\xi}^2 = 1 + \sqrt{\frac{\alpha_4 - 1}{n}}\, X - \frac{Y^2}{n}$$

where

(78) $$X = \frac{1}{\sqrt{n}} \sum \frac{(\xi_r^2 - 1)}{\sqrt{\alpha_4 - 1}}, \qquad Y = \sqrt{n}\, \bar{\xi}.$$

Hence

(79) $$G(z) = Pr\{X - \lambda Y^2 \leq z\}$$

with

(80) $$\lambda = \frac{1}{\sqrt{n(\alpha_4 - 1)}}.$$

Let W be the probability function of the distribution of the random point (X, Y) and $f(t_1, t_2)$ be the characteristic function:

(81) $$W(S) = Pr\{(X, Y) \epsilon S\} \quad \text{for every Borel set } S \text{ in } R_2,$$

(82) $$f(t_1, t_2) = \epsilon(e^{it_1 X + it_2 Y}) = \left\{p\left(\frac{t_1}{\sqrt{n}}, \frac{t_2}{\sqrt{n}}\right)\right\}^n$$

(83) $$p(t_1, t_2) = \int_{-\infty}^{\infty} e^{it_1(x^2-1)/(\sqrt{\alpha_4 - 1}) + it_2 x}\, dP.$$

Let $G_1(z)$ be the distribution function of X. Then

(84) $$G(z) - G_1(z) = \iint_{z < x \leq z + \lambda y^2} dW = K(z), \quad \text{say}.$$

Let

(85) $$K_\epsilon(z) = \iint_{z < x \leq z + \lambda y^2} e^{-\epsilon y^{2k}}\, dW.$$

If we define (for fixed z) the function $G(x, y)$ by

(86) $G(x, y) = e^{-\epsilon y^{2k}}$ if $z < x \leq z + \lambda y^2$, $\quad G(x, y) = 0$ otherwise,

then

(87) $$K_\epsilon(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, y)\, dW.$$

Letting

(88) $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-it_1 x - it_2 y} G(x, y)\, dx\, dy = g(t_1, t_2),$$

we replace x by $x - u$ in the integral and get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x-it_2y}G(x-u, y)\,dx\,dy = e^{-it_1u}g(t_1, t_2). \tag{89}$$

Multiplying both sides by $\dfrac{1-\cos Tu}{u^2}$ and integrating with respect to u we obtain, with the help of (59), Lemma 11,

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x-it_2y}\,dx\,dy\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}G(x-u, y)\,du\\ &\qquad = \begin{cases}\pi(T-|t_1|)g(t_1, t_2) & \text{if } |t_1| \le T,\\ 0 & \text{if } |t_1| > T;\end{cases}\end{aligned} \tag{90}$$

the reversion of order of integration in the left-hand side is obviously justifiable. By Lemma 9 the right-hand side of (90) is summable in the whole plane of (t_1, t_2). Hence, by Lemma 10,

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}G(x-u, y)\,du\\ &\qquad = \frac{1}{4\pi}\iint_{|t_1|\le T}(T-|t_1|)g(t_1, t_2)e^{it_1x+it_2y}\,dt_1\,dt_2.\end{aligned} \tag{91}$$

If we integrate both sides with respect to the probability function W, we obtain, on reversing the order of integration,

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}\,du\iint_{R_2}G(x-u, y)\,dW\\ &\qquad = \frac{1}{4\pi}\iint_{|t_1|\le T}(T-|t_1|)g(t_1, t_2)f(t_1, t_2)\,dt_1\,dt_2.\end{aligned} \tag{92}$$

By (86) and (87),

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x-u, y)\,dW = K_\epsilon(u+z). \tag{93}$$

Hence

$$\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}K_\epsilon(u+z)\,du = \frac{1}{4\pi}\iint_{|t_1|\le T}(T-|t_1|)g(t_1, t_2)f(t_1, t_2)\,dt_1\,dt_2. \tag{94}$$

We now take the functions

$$\varphi(t_1, t_2) = e^{-\frac{1}{2}(t_1^2+t_2^2+2\rho t_1t_2)} \tag{95}$$

and $\psi(it_1, it_2)$ as in (35), where

$$\rho = \int_{-\infty}^{\infty} \frac{(x^2 - 1)x}{\sqrt{\alpha_4 - 1}}\, dP = \frac{\alpha_3}{\sqrt{\alpha_4 - 1}}. \tag{96}$$

Since the condition $\alpha_4 - 1 - \alpha_3^2 \neq 0$ is assumed in Theorem 3 and implied in Theorem 4, we have $|\rho| < 1$. Let

$$w(x, y) = \frac{1}{2\pi\sqrt{1 - \rho^2}}\, e^{-(1/2(1-\rho^2))(x^2+y^2-2\rho xy)} \tag{97}$$

and let $\gamma(x, y)$ be the function obtained from $\psi(it_1, it_2)$ through the replacement of each power $(it_1)^{\nu_1}\ (it_2)^{\nu_2}$ by $(-1)^{\nu_1+\nu_2} W_{\nu_1\nu_2}(x, y) = (-1)^{\nu_1+\nu_2} \dfrac{\partial^{\nu_1+\nu_2} w(x, y)}{\partial x^{\nu_1} \partial y^{\nu_2}}$. Since

$$w(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-it_1x - it_2y} \varphi(t_1, t_2)\, dt_1 dt_2, \tag{98}$$

we have

$$w_{\nu_1\nu_2}(x, y) = \frac{(-1)^{\nu_1+\nu_2}}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (it_1)^{\nu_1} (it_2)^{\nu_2} e^{-it_1x - it_2y} \varphi(t_1, t_2)\, dt_1 dt_2, \tag{99}$$

whence, by Fourier inversion,

$$(it_1)^{\nu_1} (it_2)^{\nu_2} \varphi(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it_1x + it_2y} w_{\nu_1\nu_2}(x, y)\, dx dy. \tag{100}$$

From the definition of $\gamma(x, y)$ it follows therefore

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it_1x + it_2y} \{w(x, y) + \gamma(x, y)\}\, dx\, dy = \varphi(t_1, t_2)\{1 + \psi(it_1, it_2)\}. \tag{101}$$

A comparison of (101) with $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it_1x + it_2y}\, dW = f(t_1, t_2)$ shows that (94) will remain true if $K_\epsilon(u)$ be replaced by

$$\iint_{u < x \leq u + \lambda y^2} e^{-\epsilon y^{2k}} (w(x, y) + \gamma(x, y))\, dx\, dy = L_\epsilon(u), \text{ say}, \tag{102}$$

and $f(t_1, t_2)$ be replaced by $\varphi(t_1, t_2)\{1 + \psi(it_1, it_2)\}$. Hence

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} \{K_\epsilon(u + z) - L_\epsilon(u + z)\}\, du \\ &= \frac{1}{4\pi} \iint_{|t_1| \leq T} (T - |t_1|) g(t_1, t_2) \{f(t_1, t_2) \\ &\qquad - \varphi(t_1, t_2)[1 + \psi(it_1, it_2)]\}\, dt_1 dt_2. \end{aligned} \tag{103}$$

Let also

$$(104)\qquad H(z) = \iint\limits_{x-\lambda y^2 \le z} \{w(x, y) + \gamma(x, y)\}\, dx\, dy,$$

$$H_1(z) = \iint\limits_{x \le z} \{w(x, y) + \gamma(x, y)\}\, dx\, dy,$$

$$(105)\qquad L(z) = H(z) - H_1(z) = \iint\limits_{z < x \le z+\lambda y^2} \{w(x, y) + \gamma(x, y)\}\, dx\, dy.$$

3.2. We now consider the particular case $k = 3$ and prove Theorem 3. For $k = 3$ we have $\psi \equiv \gamma \equiv 0$ and so

$$(106)\qquad \begin{aligned} H(z) &= \iint\limits_{x-\lambda y^2 \le z} w(x, y)\, dx\, dy, \\ H_1(z) &= \iint\limits_{x \le z} w(x, y)\, dx\, dy = \Phi(z), \end{aligned}$$

$$(107)\qquad \begin{aligned} L(z) &= H(z) - H_1(z), \\ L_\epsilon(z) &= \iint\limits_{z < x \le z+\lambda y^2} e^{-\epsilon y^2} w(x, y)\, dx\, dy, \end{aligned}$$

$$(108)\qquad \begin{aligned} &\int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} \{K_\epsilon(u + x) - L_\epsilon(u + x)\}\, du \\ &\qquad = \frac{1}{4\pi} \iint\limits_{|t_1| \le T} (T - |t_1|) g(t_1, t_2) \{f(t_1, t_2) - \varphi(t_1, t_2)\} dt_1\, dt_2 . \end{aligned}$$

Now

$$\begin{aligned} K_\epsilon(u) - L_\epsilon(u) = \{G(u) - \Phi(u)\} - \{H(u) - \Phi(u)\} - \{G_1(u) - \Phi(u)\} \\ - \{K(u) - K_\epsilon(u)\} + \{L(u) - L_\epsilon(u)\}, \end{aligned}$$

$$\begin{aligned} 0 \le H(u) - \Phi(u) &= \frac{1}{2\pi\sqrt{1 - \rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\, dy \int_u^{u+\lambda y^2} e^{-(1/2(1-\rho^2))(x-\rho y)^2}\, dx \\ &\le \frac{\lambda}{2\pi\sqrt{1 - \rho^2}} \int_{-\infty}^{\infty} y^2 e^{-\frac{1}{2}y^2}\, dy = \frac{\lambda}{\sqrt{2\pi(1 - \rho^2)}}, \end{aligned}$$

$$|G_1(u) - \Phi(u)| \le \frac{A}{\sqrt{n}} \int_{-\infty}^{\infty} \left| \frac{x^2 - 1}{\sqrt{\alpha_4 - 1}} \right|^3 dP \le \frac{A\alpha_6}{(\alpha_4 - 1)^{3/2}\sqrt{n}} \text{ by Theorem 2,}$$

$$0 \le K(u) - K_\epsilon(u) \le \epsilon\epsilon(Y^6) \le A\alpha_6\epsilon \text{ by Lemma 12,}$$

$$0 \le L(u) - L_\epsilon(u) \le A\epsilon.$$

Hence

$$\int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} \{G(u + \lambda) - \Phi(u + \lambda)\}\, du$$

$$(109) \qquad = \theta T \left\{\alpha_6 \epsilon + \frac{\alpha_6}{(\alpha_4 - 1)^{3/2}\sqrt{n}} + \frac{1}{\sqrt{n}\sqrt{(\alpha_4 - 1)(1 - \rho^2)}}\right\} + \theta T \iint_{|t_1| \le T} |g(t_1, t_2)| \cdot |f(t_1, t_2) - \varphi(t_1, t_2)|\, dt_1 dt_2 .$$

It is easy to verify that

$$\frac{\alpha_6}{(\alpha_4 - 1)^{3/2}} + \frac{1}{\sqrt{(\alpha_4 - 1)(1 - \rho^2)}} \le \left(\frac{\alpha_6}{\alpha_4 - 1 - \alpha_3^2}\right)^{3/2}.$$

For the left-hand side of (109) we refer to (50) and take x to be the number a therein. Hence

$$T\delta\left\{3\int_0^{Ta} \frac{1 - \cos u}{u^2}\, du - \pi\right\} \le AT\left\{\alpha_6 \epsilon + \frac{1}{\sqrt{n}}\left(\frac{\alpha_6}{\alpha_4 - 1 - \alpha_3^2}\right)^{3/2}\right\}$$

$$(110) \qquad + AT \iint_{|t_1| \le T, |t_2| \le T} |g(t_1, t_2)| \cdot |f(t_1, t_2) - \varphi(t_1, t_2)|\, dt_1 dt_2$$

$$+ AT \iint_{|t_1| \le T, |t_2| > T} |g(t_1, t_2)|\, dt_1 dt_2 .$$

By Lemma 9 (ii) we have

$$T \iint_{|t_1| \le T, |t_2| > T} |g(t_1, t_2)|\, dt_1 dt_2$$

$$(111) \qquad \le AT \iint_{|t_1| \le T, |t_2| > T} \frac{1}{|t_2|^3}\left(\lambda + \frac{\lambda^2 |t_1|}{\epsilon^1} + \frac{\lambda^3 |t_1|^2}{\epsilon^1}\right) dt_1 dt_2 .$$

$$\le A\left(\lambda + \frac{\lambda^2 T}{\epsilon^1} + \frac{\lambda^3 T^2}{\epsilon^1}\right).$$

Hence

$$T\delta\left\{3\int_0^{T\delta} \frac{1 - \cos u}{u^2}\, du - \pi\right\}$$

$$(112) \qquad \le A\left\{\alpha_6 T\epsilon + \left(\frac{\alpha_6}{\alpha_4 - 1 - \alpha_3^2}\right)^{3/2} \frac{T}{\sqrt{n}} + \lambda + \frac{\lambda^2 T}{\epsilon^1} + \frac{\lambda^3 T^2}{\epsilon^1}\right\} + AT \iint_{|t_1| \le T, |t_2| \le T} |g(t_1, t_2)|\, |f - \varphi|\, dt_1 dt_2 .$$

By Lemma 9 (i) with $k = 3$ we have

$$(113) \quad T \iint\limits_{|t_1|\leq T, |t_2| \leq T} |g| \cdot |f - \varphi| \, dt_1 \, dt_2 \leq \frac{AT\lambda}{\epsilon^{\frac{1}{2}}} \iint\limits_{|t_1|\leq T, |t_2| \leq T} |f - \varphi| \, dt_1 \, dt_2 .$$

By (37) under Lemma 3,

$$(114) \quad |f - \varphi| \leq \frac{A}{\sqrt{n}} (\beta_{31} |t_1|^3 + \beta_{32} |t_2|^3) e^{-\frac{1}{4}(1-\rho^2)(t_1^2+t_2^2)} \quad \text{for } |t_i| \leq \frac{A(1-\rho^2)\sqrt{n}}{\beta_{3i}}$$

with

$$(115) \quad \begin{aligned} \beta_{31} = \int_{-\infty}^{\infty} \left| \frac{x^2 - 1}{\sqrt{\alpha_4 - 1}} \right|^3 dP &\leq \frac{4}{(\alpha_4 - 1)^{\frac{3}{2}}} \int_{-\infty}^{\infty} (x^6 + 1) \, dP \\ &\leq \frac{8\alpha_6}{(\alpha_4 - 1)^{\frac{3}{2}}}, \qquad \beta_{32} = \int_{-\infty}^{\infty} |x|^3 \, dP = \beta_3 . \end{aligned}$$

We now take

$$(116) \quad T = \frac{A}{8} \left(\frac{\alpha_4 - 1 - \alpha_3^2}{\alpha_6} \right)^{\frac{3}{2}} \sqrt{n},$$

the A coinciding with that in (114). Then

$$(117) \quad \begin{aligned} \frac{A(1 - \rho^2)\sqrt{n}}{\beta_{31}} &\geq \frac{A(1 - \rho^2)(\alpha_4 - 1)^{\frac{3}{2}}\sqrt{n}}{8\alpha_6} \\ &= \frac{A(\alpha_4 - 1 - \alpha_3^2)\sqrt{\alpha_4 - 1}\,\sqrt{n}}{8\alpha_6} \geq \frac{A(\alpha_4 - 1 - \alpha_3^2)^{\frac{3}{2}}\sqrt{n}}{8\alpha_6^{3/2}} = T \end{aligned}$$

$$(118) \quad \begin{aligned} \frac{A(1 - \rho^2)\sqrt{n}}{\beta_{32}} &= \frac{A(\alpha_4 - 1 - \alpha_3^2)\sqrt{n}}{(\alpha_4 - 1)\beta_3} \\ &\geq \frac{A(\alpha_4 - 1 - \alpha_3^2)^{\frac{3}{2}}\sqrt{n}}{\alpha_4^{3/2}\beta_3} \geq \frac{A(\alpha_4 - 1 - \alpha_3^2)^{\frac{3}{2}}\sqrt{n}}{\alpha_6^{3/2}} > T. \end{aligned}$$

Hence (114) is true for $|t_1| \leq T$ and $|t_2| \leq T$. Using this fact on (113) we obtain

$$(119) \quad \begin{aligned} T &\iint\limits_{|t_1|\leq T, |t_2| \leq T} |g||f - \varphi| \, dt_1 \, dt_2 \\ &\leq \frac{AT\lambda}{\epsilon^{\frac{1}{2}}} \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \frac{\alpha_6}{(\alpha_4 - 1)^{\frac{3}{2}}} |t_1|^3 + \beta_3 |t_2|^3 \right\} e^{-\frac{1}{4}(1-\rho^2)(t_1^2+t_2^2)} \, dt_1 \, dt_2 \\ &\leq \frac{AT\lambda}{\epsilon^{\frac{1}{2}}\sqrt{n}} \left(\frac{\alpha_6}{(\alpha_4 - 1)^{3/2}} + \beta_3 \right) \frac{1}{(1 - \rho^2)^{5/2}} \\ &= \frac{AT\lambda}{\sqrt{n\epsilon}} (\alpha_6(\alpha_4 - 1) + \beta_3(\alpha_4 - 1)^{5/2}) \frac{1}{(\alpha_4 - 1 - \alpha_3^2)^{5/2}} \\ &= \frac{AT}{n\sqrt{\epsilon}} (\alpha_6\sqrt{\alpha_4 - 1} + \beta_3(\alpha_4 - 1)^2) \frac{1}{(\alpha_4 - 1 - \alpha_3^2)^{5/2}} \\ &\leq \frac{AT\alpha_6^{11/6}}{n\sqrt{\epsilon}(\alpha_4 - 1 - \alpha_3^2)^{5/2}} . \end{aligned}$$

Substituting in (112), setting $\epsilon = (\alpha_6 T)^{-1}$ and using (116) we obtain after some easy reduction

$$
\begin{aligned}
(120)\qquad & T\delta\left\{3\int_0^{T\delta}\frac{1-\cos u}{u^2}\,du - \pi\right\} \\
& \leq A\left[1+\frac{1}{\sqrt{n(\alpha_4-1)}}+\left(\frac{\alpha_6}{n(\alpha_4-1-\alpha_3^2)}\right)^{\frac{1}{2}}+\left(\frac{\alpha_8}{n(\alpha_4-1-\alpha_3^2)}\right)^{\frac{1}{2}}\right].
\end{aligned}
$$

If $n \geq (\alpha_4 - 1 - \alpha_3^2)^{-1}\alpha_6$, then the right-hand side of (120) is $\leq A$, and so, arguing with (120), as we did with (70), we obtain

$$
(121)\qquad \text{l.u.b.}\,|G(u) - \Phi(u)| \leq \frac{A}{T} = \frac{A}{\sqrt{n}}\left(\frac{\alpha_6}{\alpha_4 - 1 - \alpha_3^2}\right)^{\frac{1}{2}}.
$$

For $n < (\alpha_4 - 1 - \alpha_3^2)^{-1}\alpha_6$, however, the right-hand side of (121) $\geq A(\alpha_4 - 1 - \alpha_3^2)^{-1}\alpha_6 \geq A$ and (121) becomes a triviality. Hence Theorem 3 is proved.

3.3. To prove Theorem 4, we start again with the identity (103). We have

$$
\begin{aligned}
(122)\qquad K_\epsilon(u) - L_\epsilon(u) = \{G(u) - H(u)\} - \{G_1(u) - H_1(u)\} \\
- \{K(u) - K_\epsilon(u)\} + \{L(u) - L_\epsilon(u)\},
\end{aligned}
$$

$$
(123)\qquad 0 \leq K(u) - K_\epsilon(u) \leq \epsilon\epsilon(Y^{2k}) \leq Q_k\epsilon \quad \text{by Lemma 12,}
$$

$$
(124)\quad 0 \leq L(u) - L_\epsilon(u) \leq \epsilon\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^{2k}(w(x, y) + |\gamma(x, y)|)\,dx\,dy \leq Q_k\epsilon.
$$

Let us show that

$$
(125)\qquad |G_1(u) - H_1(u)| \leq Q_k/n^{\frac{1}{2}(k-1)}.
$$

The function $X = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{\xi_i^2 - 1}{\sqrt{\alpha_4 - 1}}\right)$ has the same structure as $\sqrt{n}\,\bar{\xi}$ (with $(\alpha_4 - 1)^{-\frac{1}{2}}(\xi_i^2 - 1)$ playing the role of ξ_i); hence, by Theorem 1, there exists an asymptotic expansion of the distribution function $G_1(u)$. We shall see that the terms of this asymptotic expansion are precisely $H_1(u)$, whence (125) follows from Theorem 1.

It is obvious that for the polynomial $\psi(it_1, it_2)$ in (35) $\psi(it, 0)$ coincides with the polynomial $\psi(it)$ in (34). Hence the terms of the asymptotic expansion of $G_1(u)$ are the inversion of $e^{-\frac{1}{2}t^2}\{1 + \psi(it, 0)\}$ viz.

$$
(126)\qquad \Phi(u) + \frac{1}{2\pi}\int_{-\infty}^{u} dx\int_{-\infty}^{\infty} e^{-itx-\frac{1}{2}t^2}\psi(it, 0)\,dt.
$$

On the other hand, by (104),

$$
(127)\qquad H_1(u) = \Phi(u) + \int_{-\infty}^{u} dx\int_{-\infty}^{\infty}\gamma(x, y)\,dy,
$$

and by (101) with $t_2 = 0$,

$$
(128)\qquad \int_{-\infty}^{\infty} e^{itx}\,dx\int_{-\infty}^{\infty}\gamma(x, y)\,dy = e^{-\frac{1}{2}t^2}\psi(it, 0).
$$

Inversion of (118) gives

$$\int_{-\infty}^{\infty} \gamma(x, y)\, dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx-\frac12 t^2} \psi(it, 0)\, dt \tag{129}$$

which establishes the equality of $H_1(u)$ and (126).

Using (122), (123), (124), (125) on (103) we get

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} \{G(u + z) - H(u + z)\}\, du = \Lambda_k T\left(\epsilon + \frac{1}{n^{\frac12(k-2)}}\right) \\ &\qquad + \theta T \int\int |g(t_1, t_2)| \cdot |f(t_1, t_2) - \varphi(t_1, t_2)[1 + \psi(it_1, it_2)]|\, dt_1\, dt_2 . \end{aligned} \tag{130}$$

If we expand

$$H(u) = \iint_{-\lambda_n y^2 \leq u} \{w(x, y) + \gamma(x, y)\}\, dx\, dy \tag{131}$$

in powers of $n^{-\frac12}$ up to and including the term $n^{-\frac12(k-3)}$, the remainder is obviously $\Lambda_k n^{-\frac12(k-2)}$. Hence

$$H(u) = \Phi(u) + \chi(u) + \Lambda_k / n^{\frac12(k-2)}, \tag{132}$$

where $\Phi(u) + \chi(u)$ is the group of terms of the Taylor expansion of (131) in powers of $n^{-\frac12}$ up to and including the term $n^{-\frac12(k-3)}$. From (130) and (132) we get

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} \{G(u + z) - \Phi(u + z) - \chi(u + z)\}\, du \right| \\ &\qquad \leq Q_k T\left(\epsilon + \frac{1}{n^{\frac12(k-2)}}\right) + AI, \end{aligned} \tag{133}$$

where

$$I = T \iint_{|t_i| \leq T} |g(t_1, t_2)| \cdot |f(t_1, t_2) - \varphi(t_1, t_2)\{1 + \psi(it_1, it_2)\}|\, dt_1\, dt_2 . \tag{134}$$

We are going to prove that the function $\chi(u)$ here defined satisfies all the requirements of the function $\chi(u)$ in Theorem 4. The structure of $\chi(u)$ announced in Theorem 4 is easily verifiable. It remains to prove the inequalities (15) and (16) satisfied by

$$| G(u) - \Phi(u) - \chi(u) | .$$

It is obvious that the function $\Phi(u) + \chi(u)$ has all the properties of the function $F_1(u)$ in Lemma 8, having a bounded derivative $| \Phi'(u) + \chi'(u) | \leq Q_k$. Hence, on taking z in (133) to be the number a in (51), the left-hand side of (133) does not exceed

$$Q_k T\delta \left(3 \int_0^{T\delta} \frac{1 - \cos u}{u^2}\, du - \pi\right), \qquad \delta = Q_k \,\text{l.u.b.}\, | G(u) - \Phi(u) - \chi(u) |.$$

Hence

(135) $$T\delta\left(3\int_0^{T\delta}\frac{1-\cos u}{u^2}\,du-\pi\right)\le Q_k T\left(\epsilon+\frac{1}{n^{\frac{1}{2}(k-2)}}\right)+Q_k I.$$

In order to appraise I we recall (35) under Lemma 3 (replacing therein each β_{ki} by the larger number $\beta_{k1}\beta_{k2}$, and merging the latter into Q_k)

(136) $$\left|f(t_1,t_2)-\varphi(t_1,t_2)\{1+\psi(it_1,it_2)\}\right|\le\frac{Q_k}{n^{\frac{1}{2}(k-2)}}\{\Sigma(|t_i|^k+\cdots+|t_i|^{3(k-2)})\}e^{-(1-\rho^2)(t_1^2+t_2^2)/8}$$

for

(137) $$|t_i|\le Q_k\sqrt{n}.$$

Put $T=(Q_k\sqrt{n})^l$, with Q_k here coinciding with that in (137) and then (136) is valid for $|t_1|\le T^{1/l}$ and $|t_2|\le T^{1/l}$. Write

$$I=T\iint_{|t_1|\le T^{1/l},|t_2|\le T^{1/l}}+T\iint_{|t_1|\le T,|t_2|>T^{1/l}}+T\iint_{\substack{T^{1/l}<|t_1|\le T\\|t_2|\le T^{1/l}}}=I_1+I_2+I_3.$$

By Lemma 9 (i),

(138) $$I_1\le\frac{Q_kT}{n^{\frac{1}{2}}\epsilon^{3/2k}}\int\int|f-\varphi(1+\psi)|\,dt_1\,dt_2,$$

whence, by (136)

(139) $$I_1\le\frac{Q_kT}{n^{\frac{1}{2}(k-1)}\epsilon^{3/2k}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\sum_{i=1}^{2}(|t_i|^k+\cdots+|t_i|^{3(k-2)})\right)\cdot e^{-(1-\rho^2)(t_1^2+t_2^2)/8}\,dt_1\,dt_2\le\frac{Q_kT}{n^{\frac{1}{2}(k-1)}\epsilon^{3/2k}}.$$

By Lemma 9 (iii) we have

$$I_2\le Q_kT\iint_{|t_1|\le T,|t_2|>T^{1/l}}\frac{1}{|t_2|^2}\left(\frac{1}{\sqrt{n}\epsilon^{1/2k}}+\frac{|t_1|}{n\epsilon^{3/2k}}\right)\{|f(t_1,t_2)|+\varphi(t_1,t_2)\,|\,1+\psi(it_1,it_2)\,|\}\,dt_1\,dt_2.$$

Obviously,

(140) $$\underset{t_2>T^{1/k-2}}{\text{l.u.b.}}\ \varphi(t_1,t_2)\,|\,1+\psi(it_1,it_2)|=e^{-nQ_k}.$$

On the assumption of non-singularity of $P(x)$ we have, by Lemma 7,

(141) $$\underset{|t_2|>T^{1/k-2}}{\text{l.u.b.}}\ |f(t_1,t_2)|=\underset{|t_2|>Q_k\sqrt{n}}{\text{l.u.b.}}\left|p\left(\frac{t_1}{\sqrt{n}},\frac{t_2}{\sqrt{n}}\right)\right|^n=\underset{|t_2|\ge Q_k}{\text{l.u.b.}}\left|p\left(\frac{t_1}{\sqrt{n}},t_2\right)\right|^n=e^{-nQ_k}.$$

Hence

$$(142)\qquad I_2 \leq Q_k T e^{-nQ_k} \iint_{|t_1| \leq T, |t_2| > T^{1/l}} \frac{1}{|t_2|^2}\left(\frac{1}{\sqrt{n}\,\epsilon^{1/2k}} + \frac{|t_1|}{n\epsilon^{3/2k}}\right) dt_1\, dt_2 = Q_k\left(\frac{n^{l-1}}{\epsilon^{1/2k}} + \frac{n^{(3/2)(l-1)}}{\epsilon^{3/2k}}\right) e^{-nQ_k}.$$

For I_3 we have $|t_1| > T^{1/l} = Q_k\sqrt{n}$, and so Lemma 7 is applicable to I_3 in the same manner as to I_2. Using Lemma 9 (i) on the factor $|g(t_1, t_2)|$ we get

$$(143)\qquad I_3 \leq \frac{Q_k n^l e^{-nQ_k}}{\epsilon^{3/2k}}.$$

Combining (135), (138), (139), (142), (143) we obtain

$$(144)\qquad T\delta\left(3\int_0^{T\delta} \frac{1-\cos u}{u^2}\,du - \pi\right) \leq Q_k\left(n^{l/2}\epsilon + \frac{n^{l/2}}{n^{\frac{1}{2}(k-2)}} + \frac{n^{l/2}}{n^{\frac{1}{2}(k-1)}\epsilon^{3/2k}}\right) + Q_k\left(\frac{n^{l-1}}{\epsilon^{1/2k}} + \frac{n^{3/2(l-1)}}{\epsilon^{3/2k}} + \frac{n^l}{\epsilon^{3/2k}}\right) e^{-nQ_k}.$$

Putting $\epsilon = \dfrac{1}{n^{k(k-1)/(2k+3)}}$ we get, as the last term in (144) is $\leq Q_k$,

$$T\delta\left(3\int_0^{T\delta} \frac{1-\cos u}{u^2}\,du - \pi\right) \leq Q_k + Q_k n^{l/2}\left(\frac{1}{n^{k(k-1)/(2k+3)}} + \frac{1}{n^{\frac{1}{2}(k-2)}}\right).$$

If $4 \leq k \leq 6$, we take $l = k-2$ and get

$$T\delta\left(3\int_0^{T\delta} \frac{1-\cos u}{u^2}\,du - \pi\right) \leq Q_k + Q_k\left(\frac{1}{n^{(6-k)/(2(2k+3))}} + 1\right) \leq Q_k.$$

Hence, by the argument following (70),

$$\text{l.u.b.}\,|G(u) - \Phi(u) - \chi(u)| \leq \frac{Q_k}{T} = \frac{Q_k}{n^{\frac{1}{2}(k-2)}},$$

giving (15). If $k \geq 7$, we take $l = \dfrac{2k(k-1)}{2k+3}$ and get

$$T\delta\left(3\int_0^{T\delta} \frac{1-\cos u}{u^2}\,du - \pi\right) \leq Q_k + Q_k\left(1 + \frac{1}{n^{(k-6)/(2(k+3))}}\right) \leq Q_k.$$

Hence

$$\text{l.u.b.}\,|G(u) - \Phi(u) - \chi(u)| \leq \frac{Q_k}{T} = \frac{Q_k}{n^{k(k-1)/2(k+3)}},$$

giving (16). Therefore Theorem 4 is proved.

5. When $\alpha_4 - 1 - \alpha_3^2 = 0$. If $\alpha_4 - 1 - \alpha_3^2 = 0$, then there is unit probability that ξ_i assumes exactly two values:

$$Pr\{\xi_i = a\} = p, \qquad Pr\{\xi_i = b\} = q, \qquad p + q = 1.$$

Let $\zeta_i = 1$ with probability p and $\zeta_i = 0$ with probability q. Then $\xi_i = b + (a - b)\zeta_i$, $\eta = (a - b)^2 \frac{1}{n} \Sigma(\zeta_i - \bar{\zeta})^2$. Hence it is sufficient to consider the variable $\frac{1}{n} \sum (\zeta_i - \bar{\zeta})^2 = \eta_1$. Letting $\Sigma\zeta_i = r = np + \sqrt{npq}\,X$ we have $\eta_1 = r - \frac{r^2}{n} = npq + (q - p)\sqrt{npq}\,X - pqX^2$. We now consider two distinct cases:

Case (i). $p \neq q$. Here

$$F(z) = Pr\left\{\frac{\eta_1 - n/\partial q}{|p - q|\sqrt{npq}} \leq z\right\}$$

$$= Pr\{(X + c\sqrt{n})^2 \geq c^2 n - 2|c|\sqrt{n}\,z\}, \quad c = \frac{p - q}{2\sqrt{pq}}.$$

Thus $F(z) = 1$ if $z \geq \frac{1}{2}|c|\sqrt{n}$. If $z < \frac{1}{2}|c|\sqrt{n}$, then

$$F(z) = Pr\{X \leq -cn - (c^2 n - 2|c|\sqrt{n}z)^{\frac{1}{2}}\}$$
$$+ Pr\{X \geq -c\sqrt{n} + (c^2 n - 2|c|\sqrt{n}z)^{\frac{1}{2}}\} = F_1(z) + F_2(z).$$

To the random variable X Theorem 2 can be applied. Suppose that $c < 0$, then, by Tchebycheff's inequality,

$$F_2(z) \leq Pr\{X \geq -cn\} \leq \frac{1}{c^2 n} \leq \frac{1}{(p - q)^2 n}.$$

By Theorem 2,

$$F_1(z) = Pr\{X \leq -cn - (c^2 n - 2|c|\sqrt{n}\,z)^{\frac{1}{2}}\}$$
$$= \Phi(z) + \frac{Oz^2}{\sqrt{n}|p - q|} + \frac{O(p^2 + q^2)}{\sqrt{npq}}.$$

Hence

$$(145) \qquad |F(z) - \Phi(z)| \leq A\left\{\frac{p^2 + q^2}{\sqrt{npq}} + \frac{z^2}{\sqrt{n}|p - q|} + \frac{1}{n(p - q)^2}\right\}.$$

The same inequality holds also for $c > 0$.

Case (ii). $p = q = 1/2$. Here $\eta_1 = \frac{1}{4}(n - X^2)$; hence

$$(146) \quad Pr\left\{\eta_1 \geq \frac{n - z}{4}\right\} = Pr\{X^2 \leq z\} = \frac{1}{\sqrt{2\pi}} \int_0^z x^{-\frac{1}{2}} e^{-x/2}\, dx + \frac{O}{\sqrt{n}}.$$

There is no asymptotic expansion for the distribution function of η_1. (See (C), p. 83.)

SAMPLING INSPECTION PLANS FOR CONTINUOUS PRODUCTION WHICH INSURE A PRESCRIBED LIMIT ON THE OUTGOING QUALITY

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1. Introduction. This paper discusses several plans for sampling inspection of manufactured articles which are produced by a continuous production process, the plans being designed to insure that the long-run proportion of defectives shall not exceed a prescribed limit. The plans are applicable to articles which can be classified as "defective" or "non-defective" and which are submitted for inspection either continuously or in lots. In Section 2 the notions of "average outgoing quality limit" and "local stability" are discussed. The valuable concept of average outgoing quality limit for lot inspection is due to Dodge and Romig [4], and that for inspection of continuous production to Dodge [1]. Section 3 contains a description of a simple inspection plan (SPA) applicable to to continuous production and a proof that the plan will insure a prescribed average outgoing quality limit. Section 4 contains a proof that this inspection plan also has the important property that it requires minimum inspection when the production process is in statistical control. In Section 5 is contained the description of a general class of plans which possess both these important properties.

The problem of adapting SPA to the case when the articles are submitted for inspection in lots instead of continuously, is treated in Section 6. Some methods of achieving local stability are discussed in Section 7 and a specific plan is developed there. Finally Section 8 discusses the relationship between the present work and that of the earlier and very interesting paper of H. F. Dodge [1], mentioned above.

If a quick first reading is desired the reader may omit the second half of Section 3 (which contains a proof of the fact that SPA guarantees the prescribed average outgoing quality limit) and the entire Section 4 except for its title (the proof of the statement made in the title of Section 4 occupies the whole section).

2. Fundamental notions. In this paper we shall deal only with a product whose units can be classified as "defective" or "non-defective." We shall assume that the units of the product are submitted for inspection continuously, except in Section 6, where we assume that they are submitted in lots. Throughout the paper we shall assume that the inspection process is non-destructive, that it invariably classifies correctly the units examined, and that defective units, when found, are replaced by non-defectives. By the "quality" of a sequence of units is meant the proportion of defectives in the sequence as produced. By the "outgoing quality" (OQ) of a sequence is meant the proportion of defectives after whatever inspection scheme which is in use has been applied. If this scheme involves random sampling, then in general the OQ is a chance variable.

(It depends on the variations of random sampling.) If the OQ converges to a constant p_a with probability one as the number of units produced increases indefinitely, p_a is called the "average outgoing quality" (AOQ). The AOQ when it exists is therefore the average quality, in the long run, of the production process after inspection. It is a function of both the production process and the inspection scheme. These definitions are due to Dodge [1].

The "average outgoing quality limit" (AOQL) is a number which is to depend only on the inspection scheme and not at all on the production process. Roughly speaking, it is a number, characteristic of an inspection scheme, such that no matter what the variations or eccentricities of the production process, the AOQ never exceeds it. For the purposes of this paper we shall need the following precise definition: Let c_i be zero or one according as the ith unit of the product, before application of the inspection scheme, is a non-defective or a defective, respectively. Let d_i have a similar definition *after* application of the inspection scheme. (We note that if the ith item was inspected, then $d_i = 0$; if the ith item was not inspected, then $c_i = d_i$.) The sequence $c = c_1, c_2, \cdots, c_N, \cdots,$ ad inf. characterizes the production process[1]. The elements of $d = d_1, d_2, \cdots,$ ad inf. are in general chance variables. The number L is called the AOQL if it is the smallest[2] number with the property that the probability is zero that

$$\limsup_N \frac{\sum_{i=1}^{N} d_i}{N} > L,$$

no matter what the sequence c.

It should be noted that this definition of AOQL places no restrictions whatever on the production process, since *all* sequences c are admitted. It is too much to expect a production process to remain always in control; indeed, doubt as to whether statistical control always exists may cause a manufacturer to institute an inspection scheme. The inspection schemes which we shall give below will yield a specified AOQL no matter what the variations in production are. If these schemes are employed, then, even if Maxwell's demon of gas theory fame were to transfer his activities to the production process, he would be unsuccessful in an effort to cause the AOQL to be exceeded. A dishonest manufacturer might sometimes essay to do this. If we imposed restrictions on the sequence c and

[1] This use of an infinite sequence to describe the production process deserves a few words. What we consider in this paper are schemes applicable when the number of units produced is large and operate mathematically as if the production sequence were of infinite length. Naturally the latter is never the case in actuality. However, the larger the number of units produced the more nearly will the reality conform to the results derived from the mathematical model. While the present definition uses explicitly the notion of an infinite sequence, such a commonplace statement as "the probability is 1/2 that a coin will fall heads up" uses this notion implicitly. It is also implicit in the intuitive meaning we ascribe to such a word as "average," which is in every day use.

[2] It is not difficult to see that such a number always exists, for it is the lower bound of a set which is non-empty (it contains the point one), bounded from below (zero is a lower bound), and closed.

determined the AOQL on that basis, we would run the danger that the relative frequency of defects in the sequence of outgoing units might exceed the AOQL if it happened that the actual sequence c did not satisfy the restrictions imposed.

After we discuss below various possible sampling inspection plans which insure that the AOQL does not exceed a predetermined value L, it will be seen that for any given $L > 0$ there are many sampling inspection schemes which do this. To choose a particular sampling plan from among them the following considerations may be advanced: If two inspection plans S and S' both insure the inequality AOQL $\leq L$ and if for any sequence c the average number of inspections required by S is not greater than that required by S' and if for some sequences c the average number of inspections required by S is actually smaller than that required by S', then S may be considered, in general, a better inspection plan than S'. However, the amount of inspection required by a sampling plan is not always the *only* criterion for the selection of a proper sampling scheme. There may be also other features of a sampling plan which make it more or less desirable. We shall mention here one such feature, called "local stability," which will play a role in our discussions later. Consider the sequence d obtained from the sequence c by applying a sampling inspection scheme. Even if the AOQL does not exceed L, it may still happen that there will be many large segments of the sequence d within which the relative frequency of ones is considerably higher than L. For instance, it may happen that in the segment $(d_1, \cdots, d_m)$ the relative frequency of ones is equal to $\frac{3}{2}L$, in the segment $(d_{m+1}, \cdots, d_{2m})$ the relative frequency is equal to $\frac{1}{2}L$, in the segment $(d_{2m+1}, \cdots, d_{3m})$ the relative frequency is again equal to $\frac{3}{2}L$, and this is followed again by a segment of m elements where the relative frequency of ones is equal to $\frac{1}{2}L$, and so forth. If m is large, such a sequence d is not very desirable, since each second segment will contain too many defects. A sequence d is said to be not locally stable if there exists a large fixed integer m such that the relative frequency of ones in $(d_{k+1}, \cdots, d_{k+m})$ is considerably greater than L for *many* integral values k. On the other hand, the sequence d is said to be locally stable if for any large m the relative frequency of ones in $(d_{k+1}, \cdots, d_{k+m})$ is not substantially above L for nearly all integral values k. This is clearly not a precise definition of "local stability," but merely an intuitive indication of what we want to understand by the term, since we did not define what we mean by "large m," "many values of k," "considerably above L," etc. A precise definition of local stability will not be needed in this paper, since it is not our intention to develop a complete theory for the choice of the sampling plan. The idea of local stability will be used in this paper merely for making it plausible that some schemes we shall consider behave reasonably in this respect. A similar idea, called "protection against spotty quality," is discussed by Dodge [1]. A possible precise definition of local stability could be given in terms of the frequency with which $F(N) = \frac{1}{(k+1)} \sum_{i=N}^{N+k} d_i$ (k being fixed) lies within given limits.

3. A sampling inspection plan which insures a given AOQL no matter what the variations in the production process. The only feature of the sampling (inspection) plan (SP) studied in this section and hereafter referred to as SPA which we shall consider here is that it insures the achievement of a specified AOQL. Considerations leading to a choice among several schemes are postponed to later sections.

For convenience, let f be the reciprocal of a positive integer. SPA calls for alternating partial inspection and complete inspection. Partial inspection is performed by inspecting one element chosen at random from each of successive groups of $\frac{1}{f}$ elements. Complete inspection means the inspection of every element in the order of production SPA is completely defined when a rule is given for ending one kind of inspection and beginning the other.

It is clear that all SP need not be of the above class. Thus, for example, a scheme might consist of partial inspection with various f's employed in various sequences. We make no attempt in this paper to examine all possible schemes. For simplicity in practical operation, alternation of complete inspection and partial inspection with fixed f would seem reasonable. The Dodge scheme [1] is of this type.

We shall also not discuss the question of a choice of the constant f, but will assume that a particular value has been chosen for various reasons and is a datum of our problem. Reasons which might influence a manufacturer in his choice of f could be contract specifications which impose a minimum on the amount of inspection, or psychological grounds to the same effect. The manufacturer may desire a certain minimum amount of inspection in order to detect malfunctioning of his production process. Also f controls local stability to some extent. The consequences of a choice of f as they appear in the theory below may also play a role

Returning to SPA, we begin with partial inspection. Let L be the specified AOQL. Denote by k_N the number of groups of $\frac{1}{f}$ units in which defectives were found as the result of *partial* inspection from the beginning of production through the Nth unit. SPA is as follows:

(a) Begin with partial inspection.

(b) Begin full inspection whenever

$$e_N = \frac{k_N\left(\frac{1}{f} - 1\right)}{N} > L.$$

(c) Resume partial inspection when

$$e_N \leq L.$$

(d) Repeat the procedure. (It will be recalled that defective units, when found, are always to be replaced with non-defectives.)

It is to be observed that in this plan the number of partial inspections increases without limit. For, while complete inspection is going on, the value of k_N remains constant, so that after a long enough period of complete inspection the denominator N of the expression which defines e_N will have increased sufficiently for e_N to be not greater than L. On the other hand, complete inspection may never occur. This will be the case if, for example, no defectives or very few defectives are produced.

We shall now show that the AOQL of the above SP is L. We first note that, at N, e_N can increase only by $\dfrac{\left(\frac{1}{f} - 1\right)}{N}$. Hence, for sufficiently large N, $e_N < L + \epsilon$, where $\epsilon > 0$ may be arbitrarily small.

Suppose now that the production process is subject to any variations whatsoever, i.e., the sequence

$$c = c_1, c_2, \cdots, c_N, \cdots, \text{ad inf.}$$

is any arbitrary sequence whatever (by their definition the c_i are all zero or one). Our result is therefore proved if we show that, with probability one,

$$\lim_{N\to\infty}\left(e_N - \frac{1}{N}\sum_{i=1}^{N} d_i\right) = 0 \tag{3.1}$$

for this arbitrary c, and that for at least one c

$$\lim_{N\to\infty} e_N = L. \tag{3.2}$$

Let $S(N)$ be the number of groups of $\frac{1}{f}$ units which have been partially inspected through the Nth unit. Define x_i as zero if in the ith partially inspected group a non-defective was found and as one if a defective was found. We have

$$k_N = \sum_{i=1}^{S(N)} x_i .$$

Since the number of times partial inspection takes place increases indefinitely, $S(N) \to \infty$ as $N \to \infty$. Also $S(N) \leq fN < N$. Let α_j be the serial number of the last unit in the jth partially inspected group. Then for all j the expected value $E(x_j)$ of x_j is given by

$$E(x_j) = f\left(\sum_{i=(\alpha_j-(1/f)+1)}^{\alpha_j} c_i\right).$$

We have, for all j

$$\sum_{i=(\alpha_j-(1/f)+1)}^{\alpha_j} (c_i - d_i) = x_j \tag{3.3}$$

so that

$$E\left(\left[\frac{1}{f} - 1\right]x_j - \sum_{\alpha_j-(1/f)+1}^{\alpha_j} d_i\right) = 0.$$

Also from (3.3) it follows, since x_j is the value of a binomial chance variable from a population of fixed number $\left(\frac{1}{f}\right)$, that there exists a positive constant β such that

$$\sigma^2\left(\left[\frac{1}{f} - 1\right]x_j - \sum_{\alpha_j - (1/f)+1}^{\alpha_j} d_i\right) < \beta \tag{3.4}$$

where $\sigma^2(x)$ is the variance of a chance variable x. Now a theorem of Kolmogoroff (Kolmogoroff [2], Fréchet [3], p. 254) states:
A sequence of chance variables with zero means and variances $\sigma_1^2, \sigma_2^2, \cdots$ converges with probability one towards zero in the sense of Césaro if

$$\sum_{i=1}^{\infty} \frac{\sigma_i^2}{i^2} \tag{3.5}$$

converges. The inequality (3.4) permits us to apply this theorem to the sequence of chance variables of which the jth ($j = 1, 2, \cdots$ ad inf.) is

$$\left(\left[\frac{1}{f} - 1\right]x_j - \sum_{\alpha_j - (1/f)+1}^{\alpha_j} d_i\right),$$

since the series $\sum_{i=1}^{\infty} \frac{1}{i^2}$ is well known to be convergent. We therefore obtain that, with probability one,

$$\lim_{S(N)\to\infty} \frac{\left(\left[\frac{1}{f} - 1\right]\sum_{j=1}^{S(N)} x_j - \sum_{j=1}^{N} d_j\right)}{S(N)} = \lim_{N\to\infty} \frac{N}{S(N)}\left(e_N - \frac{1}{N}\sum_{i=1}^{N} d_i\right) = 0,$$

since the units which are fully inspected contribute nothing to Σd_i. Since $S(N) < N$, the desired result (3.1) is a fortiori true.

If c is such that all the c_i are one, it is readily seen that (3.2) holds. If many (this adjective can be precisely defined) defectives are produced, this will also be the case. This completes the proof of the fact that the AOQL of SPA is L no matter how capriciously the production process may vary.

4. When the production process is in statistical control, SPA requires minimum inspection. The production process is said to be in statistical control if there is a positive constant $p \leq 1$ such that, for every i, the probability that $c_i = 1$ is p and is independent of the values taken by the other c's. We shall see that if the process is in statistical control and if SPA is applied to it, the specified AOQL is guaranteed with a minimum amount of inspection.

The number of units inspected through the Nth unit produced is

$$I(N) = N - \left(\frac{1}{f} - 1\right)S(N). \tag{4.1}$$

If the process is in statistical control we have, with probability one,

$$\lim_{N \to \infty} \frac{\sum_{i=1}^{N} c_i}{N} = p \tag{4.2}$$

by the strong law of large numbers. Shortly we shall prove the existence of a constant L^* such that, with probability one,

$$\lim_{N \to \infty} \frac{\sum_{i=1}^{N} d_i}{N} = L^*. \tag{4.3}$$

Assume for the moment that this is so. Since it is only by inspection that defectives are removed, and the units selected for inspection are in statistical control like the original sequence, it follows that, with probability one,

$$\lim_{N \to \infty} \frac{I(N)}{N} = \frac{1}{p}(p - L^*) = 1 - \frac{L^*}{p} \tag{4.4}$$

because, with probability one,

$$\lim_{N \to \infty} \frac{\sum_{i=1}^{N} (c_i - d_i)}{N} = p - L^*.$$

Inspection is therefore at a minimum when L^* is at a maximum compatible with the specified AOQL. By (4.3) the latter means that

$$L^* \leq L. \tag{4.5}$$

SPA has been shown to guarantee this requirement. The optimum situation from the point of view of the amount of inspection would therefore be to have $L^* = L$, but this cannot always be achieved. The absolute minimum amount of inspection clearly is f, i.e., partial inspection exclusively. Consequently from (4.4)

$$1 - \frac{L^*}{p} \geq f$$

so that

$$L^* \leq p(1 - f). \tag{4.6}$$

Combining (4.5) and (4.6) we see that we have to consider three cases:

Case a. If

$$p > \frac{L}{1 - f} \tag{4.7}$$

we have to show that

$$L = L^*. \tag{4.8}$$

Case b. If

$$p < \frac{L}{1 - f} \tag{4.9}$$

we have to show, by (4.4), that

$$1 - \frac{L^*}{p} = f,$$

that is,

(4.10) $$L^* = p(1 - f)$$

Case c. If

(4.11) $$p = \frac{L}{1 - f}$$

we have to show that

(4.12) $$L = L^* = p(1 - f).$$

PROOF of (4.8): We have already remarked in Section 3 that in SPA partial inspection always recurs, but complete inspection need never occur. We shall show in a moment that (4.7) implies that no matter how large an integer γ is chosen, the probability of temporarily stopping partial inspection for some $N > \gamma$ is one. Assume that this is so. Choose an arbitrarily small positive ϵ, and let $\gamma > \dfrac{\left(\frac{1}{f} - 1\right)}{\epsilon}$. For a sequence where complete and partial inspection alternate infinitely many times let

$$A = \alpha_1, \alpha_2, \cdots, \text{ad inf.}$$

be the sequence of integers at which partial inspection ends, and let

$$B = \beta_1, \beta_2, \cdots, \text{ad inf.}$$

be the sequence of integers at which complete inspection ends. Then, for all j,

$$\alpha_{j+1} > \beta_j > \alpha_j .$$

From the description of SPA it follows that, for all $N > \gamma$ which belong to either A or B,

(4.13) $$| e_N - L | < \epsilon.$$

In Section 3 we proved

(3.1) $$\lim_{N \to \infty} \left(e_N - \frac{1}{N} \sum_{i=1}^{N} d_i \right) = 0$$

with probability one. Since ϵ is arbitrarily small it follows that, with probability one,

(4.14) $$\lim_{\substack{N \to \infty \\ (N \text{ in } A \text{ or } B)}} \frac{\sum_{i=1}^{N} d_i}{N} = L.$$

To complete the proof of (4.8) we have still to show that L^* exists and that the probability is one that complete inspection will occur infinitely many times. First we prove that L^* exists.

As N increases during an interval of complete inspection, $D(N) = \sum_{i=1}^{N} d_i$ remains constant. Hence $\frac{D(N)}{N}$ decreases monotonically. Since for the ends of such intervals (4.14) holds, it follows that (4.14) holds as $N \to \infty$ and is a member of A, B, or an interval (α_j, β_j) for all j.

Let $N \to \infty$ while always being in the interior of an interval $(\beta_j, \alpha_{j+1}]$, $j = 1, 2, \cdots$, ad inf., which contains α_{j+1} but not β_j. Let N^* be the total number of units in these intervals through the Nth unit produced. Let N_1 and N_2 be such that

$$\beta_j = N_1 < N_2 < \alpha_{j+1}.$$

Then

$$N_2^* - N_1^* = N_2 - N_1.$$

Since the production process is in statistical control, we have, by the strong law of large numbers,

$$\lim_{N\to\infty} \frac{D(N)}{N^*} = p(1 - f) = p' \tag{4.15}$$

with probability one. Let δ^* be the general designation for numbers $<\epsilon$ in absolute value, so that all δ^* are not the same. With probability one for almost all N, we have by (4.15)

$$\frac{D(N_1)}{N_1^*} = p' + \delta^*$$

$$\frac{D(N_2)}{N_2^*} = p' + \delta^*.$$

Write

$$\frac{[D(N_2) - D(N_1)]}{(N_2 - N_1)} = K.$$

Now

$$\frac{D(N_2)}{N_2^*} = \frac{D(N_1) + [D(N_2) - D(N_1)]}{N_1^* + (N_2^* - N_1^*)} = \frac{D(N_1) + [D(N_2) - D(N_1)]}{N_1^* + (N_2 - N_1)}$$

$$= \frac{(p' + \delta^*)N_1^* + K(N_2 - N_1)}{N_1^* + (N_2 - N_1)} = p' + \delta^*.$$

Hence

$$K(N_2 - N_1) = 2\delta^* N_1^* + (p' + \delta^*)(N_2 - N_1). \tag{4.16}$$

Now suppose (4.3) does not hold. From the definition of AOQL it follows that for some $\eta > \epsilon$ there exist sequences (whose totality has a positive probability) so that, for infinitely many N_2 we have

$$\frac{D(N_2)}{N_2} = \frac{D(N_1) + [D(N_2) - D(N_1)]}{N_1 + (N_2 - N_1)} < L - 4\eta. \tag{4.17}$$

For large enough N_1, from (4.14),

$$\frac{D(N_1)}{N_1} = L + \delta^*$$

with probability one and hence, using (4.16) in (4.17)

$$\begin{aligned} N_1(L + \delta^*) + 2\delta^* N_1^* + (p' + \delta^*)(N_2 - N_1) \\ < LN_1 + L(N_2 - N_1) - 4\eta N_2 \end{aligned} \tag{4.18}$$

from which, using the fact that $p' \geq L$ (from (4.7)), we get

$$N_1\delta^* + 2N_1^*\delta^* + \delta^*(N_2 - N_1) < -4\eta N_2. \tag{4.19}$$

((4.18) and (4.19) hold for the sequences for which (4.17) holds, except perhaps on a set of sequences whose probability is zero.) Since $N_1^* \leq N_1$ and $|\delta^*| < \eta$, we have, on the other hand,

$$\begin{aligned} N_1\delta^* + 2N_1^*\delta^* + \delta^*(N_2 - N_1) &\geq -3\eta N_1 - \eta(N_2 - N_1) \\ &> -4\eta N_1 - 4\eta(N_2 - N_1) = -4\eta N_2 \end{aligned} \tag{4.20}$$

which contradicts (4.19) and proves the desired result ((4.3) and (4.8)), except that it remains to prove that, no matter how large γ, the probability of temporarily stopping partial inspection at some $N > \gamma$ is one. Let $\gamma_0 \geq \gamma$ be some integer at which partial inspection is going on. From (4.2) and (4.7) it would follow, if partial inspection never ceased on a set of sequences with positive probability, that, on this set, with conditional probability one, for N sufficiently large and ϵ sufficiently small,

$$\frac{k_N - k_{\gamma_0}}{f(N - \gamma_0)} > \frac{L}{1 - f} + \epsilon,$$

$$\frac{N}{N - \gamma_0} \frac{k_N(1 - f)}{fN} > L + (1 - f)\epsilon,$$

$$e_N > L\frac{N - \gamma_0}{N} + \frac{(N - \gamma_0)(1 - f)\epsilon}{N},$$

$$e_N > L + \frac{(1 - f)\epsilon}{2}.$$

This contradiction proves that complete inspection is eventually resumed and completes the proof of minimum inspection in Case a.

PROOF of (4.10): We shall prove that (4.9) implies that, with probability one, complete inspection will cease, never to be resumed. For, from (4.15) and (4.9) it follows that for N sufficiently large and ϵ sufficiently small,

$$\frac{D(N)}{N^*} = p' + \delta^* < L - 2\epsilon. \tag{4.21}$$

Hence, a fortiori,

$$\frac{D(N)}{N} < L - 2\epsilon. \tag{4.22}$$

((4.21) and (4.22) hold with probability one.)
(3.1) states that, with probability one,

$$\lim_{N\to\infty}\left(e_N - \frac{D(N)}{N}\right) = 0.$$

Hence for all N sufficiently large, with probability one,

$$e_N < L - \epsilon,$$

i.e., with probability one complete inspection is never resumed.

When (4.9) holds, therefore, with probability one and with a finite number of exceptions SPA will require only partial inspection.

PROOF of (4.12): If $p = \frac{L}{1-f}$ and complete inspection finally never resumes, then (4.12) follows easily. If $p = \frac{L}{1-f}$ and partial and complete inspection alternate infinitely many times, then the proof is similar to that of (4.8) and is therefore omitted. In either case the desired result follows.

5. A class of SP all of which insure both a given AOQL and minimum inspection. Let the definition of SPA be modified in the following particulars:

(b) Begin full inspection whenever

$$e_N = \frac{k_N\left(\frac{1}{f} - 1\right)}{N} > L + \phi(N).$$

(c) Resume partial inspection when

$$e_N \leq L - \psi(N).$$

Let $\phi(N)$ and $\psi(N)$ be such that

$$-\psi(N) \leq \phi(N)$$

$$\lim_{N\to\infty}\phi(N) = \lim_{N\to\infty}\psi(N) = 0.$$

(SPA corresponds to the case $\phi(N) \equiv \psi(N) \equiv 0$.) Then all the SP of this class have the property that the AOQL is L and that inspection is at a minimum in

the sense of Section 4. The proofs are essentially the same as those for SPA and hence will be omitted.

6. The inspection plans of Section 5 can also be applied to lot inspection. We shall carry on the discussion of this section in terms of SPA, but the results apply to all the members of the class of plans described in Section 5. We shall show that SPA can also be applied when the product is submitted for inspection in lots. Although we assumed previously that the units of the product are arranged in order of production, the results obtained for SPA remain valid for any arbitrary arrangement of the units. If the product is submitted in lots we may arrange the units as follows: Let l_1, l_2, $\cdots$, etc. be the successive lots in the order of their submission for inspection. Within each lot we consider the units arranged in the order in which they are chosen for inspection. In this way we have arranged all units in an ordered sequence and the inspection can be applied as described before. Thus, we start with partial inspection, i.e., we take out groups of $\frac{1}{f}$ elements in l_1 and inspect one unit (selected at random) from each of these groups. When $e_N > L$, we start complete inspection and revert to partial inspection as soon as $e_N \leq L$. When the units in l_1 are used up in the process of inspection, we continue, using the units of l_2, etc.

If it is found inconvenient to take out a group of $\frac{1}{f}$ units and then to select one unit for inspection, we could modify the sampling inspection plan as follows: Instead of taking out a group of $\frac{1}{f}$ units and then selecting at random one unit from it, we select at random *one* unit from the uninspected part of the lot and look upon this unit as the unit selected at random from a hypothetical group of $\frac{1}{f}$ units. Thus we can proceed exactly as before, except that we have to keep in mind that with each unit inspected under "partial inspection" we have used up another set of $\frac{1}{f} - 1$ units Thus, as soon as $\left(\frac{1}{f} - 1\right)$ times the number of units inspected under "partial inspection" becomes equal to or greater than the number of units in the uninspected part of the lot, the inspection of that lot is already terminated, and we have to start using the units of the next lot. The inconvenience caused by the necessity of keeping track of the number of units inspected under "partial inspection" and of the number of units in the uninspected part of the lot can be eliminated by further modifying the inspection plan as follows: Instead of beginning complete inspection as soon as $e_N > L$, we continue "partial inspection" until $E_N = e_N - L$ is so large that complete inspection of all the units of the lot not yet used up has to be made in order to bring e_N down to L at the end of the lot. This leads to the following sampling procedure, to be known as SPB: Let N_0 be the number of units in the lot, let N_L be the serial number of the last unit in the preceding lot, and let $E(N_L) =$

$N_L E_{N_L} = N_L(e_{N_L} - L)$ be the "excess" carried over from the preceding lot. For simplicity assume that the following are all integers.

$$LN_0 = M$$

$$\frac{fM}{1-f} = M^*$$

$$fN_0 = N^*$$

and

$$\frac{fE(N_L)}{1-f} = E^*.$$

The inspection procedure is then as follows: Inspect successive units drawn at random until either

(a) $M^* - E^*$ defectives have been found in the first $N' < N^*$ units inspected. In this case inspect further an additional $N_0 - \frac{N'}{f}$ units and this terminates the inspection of the lot. The excess to be carried over to the next lot is then zero.

Or

(b) N^* units have been inspected and the number of defectives found is $H \leq M^* - E^*$. In this case the inspection of the lot is terminated and the present negative excess

$$E(N_L + N_0) = [H - (M^* - E^*)] \frac{(1-f)}{f}$$

is carried over to the next lot. (The serial number of the last element in the present lot is $N_L + N_0$ and

$$e_{(N_L+N_0)} = \frac{N_L e_{N_L} + H \dfrac{(1-f)}{f}}{N_L + N_0}.$$

Hence the present excess is

$$\begin{aligned}(N_L + N_0)[e_{(N_L+N_0)} - L] &= N_L e_{N_L} + H \frac{(1-f)}{f} - LN_L - LN_0 \\ &= N_L(e_{N_L} - L) + H \frac{(1-f)}{f} - M \\ &= \frac{(1-f)}{f} [H - M^* + E^*],\end{aligned}$$

as given above.)

We note an important property of SPB: The excess carried over from a preceding lot is never positive.

7. Possible modifications of the SP to achieve local stability. Although the sampling plans discussed in previous sections are optimum in the sense that they guarantee the desired AOQL with a minimum of inspection when the production process is in statistical control, they do not always behave very favorably as far as local stability is concerned. To make this point clear, consider the following example: Suppose that during a very long initial time period the production process functions very well and the relative frequency of defectives produced is well below L. Thus, applying SPA, say, $e_N - L$ will be considerably less than zero at the end of this period. Now suppose that then the production process suddenly deteriorates and the number of defectives produced during the next period of time is considerably higher than L. In spite of that, complete inspection will not begin for quite some time because e_N became so small during the initial period. Thus there will be a long segment in the sequence of outgoing units within which the relative frequency of defectives will be larger than the prescribed AOQL. Of course, this segment will be counterbalanced by other segments where the relative frequency of defectives will be below the AOQL, so that the AOQL will not be violated. Nevertheless, the occurrence of long segments with too many defectives, i.e., a lack of local stability, is not desirable.

It should be noted that, even though SPA was not designed to achieve considerable local stability, drastic lack of local stability cannot occur when the production process is in statistical control and SPA is employed. In the example given above where the outgoing quality was not locally stable, it was assumed that there were variations in the production process. The existence of statistical control acts as an important stabilizing factor on the quality.

In this section we want to discuss several possible modifications of SPA which will insure a greater degree of local stability. One such modification is the following: We choose a positive constant A and we define the excess E_N^* for each value N as follows: $E^*(N)$ is equal to the excess $E(N)$ as originally defined $(= N[e_N - L])$ as long as for all $N' \leq N$, $E(N') \geq -A$. The difference $E^*(N+1) - E^*(N) = E(N+1) - E(N)$ for all N for which $E(N+1) - E(N) \geq 0$. If $E(N+1) - E(N) < 0$, then $E^*(N+1) = \max\,[E^*(N) + \{E(N+1) - E(N)\}, -A]$. In other words, with this modification of the sampling inspection plan we set a lower bound $-A$ for the excess. When the excess is positive we begin complete inspection, and revert to partial inspection when the excess becomes non-positive. The effect of this is that, if the proportion of defectives produced becomes large, complete inspection will not be delayed very long, although the proportion of defectives produced in the preceding period may have been considerably below L. It is clear that this modification of SPA does not increase the AOQL. However, the amount of inspection will be somewhat increased, especially when the quality of the product is less than or only slightly greater than L. If the constant A is large, the increase in the amount of inspection is only slight, but also the degree of local stability achieved is not very high. On the other hand, if A is small, the increase

in the amount of inspection may be considerable, but a high degree of local stability is achieved Thus, the choice of A should be made so that a proper balance between local stability and amount of inspection is achieved.

Modifying SPA by setting a lower limit for the excess has the disadvantage that the mathematical treatment of this case is involved. We shall, therefore, consider another modification of the inspection plan which will have largely the same effect, but whose mathematical treatment appears to be much simpler. A fixed positive integer N_0 is chosen and the inspection scheme is designed so that $E_{N_0} \leq 0$ is assured. If E_{N_0} is negative, we replace it by zero. In other words, no excess is carried over from the first segment of N_0 units to the next segment of N_0 units. Thus, the second segment of N_0 units is treated exactly the same way as if it were the first segment, and this is repeated for each consecutive segment of N_0 units This modification of SPA (the resulting plan is to be known as SPC) has essentially the same effect as setting a lower bound for the excess. Again it is clear that by this modification the AOQL is not increased, but the amount of inspection may be increased. The latter is particularly true when N_0 is small, which corresponds to very high local stability requirements. More efficient plans than SPC can probably be devised for this situation.

Undoubtedly, there are many other possible modifications of the inspection plan by which a greater degree of local stability can be achieved at the price of somewhat increased inspection. It is not the purpose of this paper to enumerate all these possibilities or to develop a theory as to which of them may be considered an optimum procedure. We shall restrict ourselves to a discussion of the mathematical consequences of SPC. First we define it precisely. If it is to be applied to inspection of lots of size N_0 then SPC is simply SPB with $E(N_L)$ and E^* always zero. When applied to continuous production it will operate as follows: Assume for convenience that $M = LN_0$, $N^* = fN_0$, and $\frac{fM}{1-f} = M^*$ are all integers.

(a) Begin each segment of N_0 units with partial inspection, i.e., inspect one unit chosen at random from each successive group of $\frac{1}{f}$ units. Continue partial inspection until one of the following events occurs: either

(b) M^* defectives are found. In this case begin complete inspection with the first unit which follows the group in which the last of the M^* defectives was found and continue until the end of the segment of N_0 units.

or

(b′) N^* groups of $\frac{1}{f}$ units are partially inspected.

(c) Repeat with the next segment of N_0 units.

Comparison with SPB shows that, in SPC, if (b) occurs earlier or at the same time as (b′), then $E_{N_0} = 0$, while if (b′) occurs before (b) we have $E_{N_0} < 0$. In contradistinction to SPB, in SPC there is no carrying over of the excess.

Let us determine the AOQ for SPC when the production process is in a state

of statistical control. Denote by p the probability that a unit produced will be defective. Let the chance variable H denote the number of defectives found during partial inspection. The probability that $H = i < M^*$ is

$$\binom{N^*}{i} p^i (1 - p)^{N^*-i}.$$

$H \leq M^*$ always. We have, when $H = i$,

$$E(N_0) = \frac{(1 - f)i}{f} - LN_0,$$

and hence

$$N_0 e_{N_0} = \frac{(1 - f)i}{f}.$$

The AOQ is therefore $\frac{(1 - f)}{fN_0}$ multiplied by the expected value of H and is therefore

$$\text{(7.1)} \quad \begin{aligned} &\frac{(1 - f)}{fN_0}\left[M^* - \sum_{i=0}^{M^*-1} (M^* - i)\binom{N^*}{i} p^i(1 - p)^{N^*-i}\right] \\ &\qquad = L\left[1 - \frac{1}{M^*}\sum_{i=0}^{M^*-1} (M^* - i)\binom{N^*}{i} p^i(1 - p)^{N^*-i}\right]. \end{aligned}$$

The reduction from the original quality p to the AOQ was achieved by inspecting a fraction of units which is $\frac{1}{p}$ times the reduction in the frequency of defectives. Hence, with probability one, the fraction of units inspected when the production process is in statistical control is

$$\text{(7.2)} \qquad I = 1 - \frac{L}{p} + \frac{(1 - f)}{pN^*}\sum_{i=0}^{M^*-1} (M^* - i)\binom{N^*}{i} p^i(1 - p)^{N^*-i}.$$

When $p \geq \frac{L}{1 - f}$ we see from Section 4 that the third term of the right member of (7.2) represents the price paid in fraction of inspection above the minimum in return for the local stability achieved. When $p < \frac{L}{1 - f}$ the additional inspection is of course $I - f$.

As N_0 becomes larger, SPC becomes more and more like SPA, and consequently the amount of inspection tends to the minimum. As N_0 becomes smaller, the degree of local stability achieved becomes higher and must be paid for by an increasing amount of inspection. An illustrative example will be given in the next section. It has already been pointed out that the mere existence of statistical control implies a considerable amount of local stability even when SPA is applied.

The only practical difficulty which may arise in evaluating the formulas in (7 1) and (7.2) might come from attempting to evaluate

$$T' = \sum_{i=0}^{M^*-1} (M^* - i) \binom{N^*}{i} p^i (1 - p)^{N^*-i}.$$

For those values of the parameters which are likely to occur in application, a good approximation to T' (exactly how good we shall not investigate here) is given by

$$T = \sum_{i=0}^{M^*-1} (M^* - i) \frac{e^{-N^*p}(N^*p)^i}{i!} .$$

A table of T for integral values of M^* from 2 to 16 and for integral values of N^*p from 1 to 25 is given below. The computations were performed under the direction of Mr. Mortimer Spiegelman of the Metropolitan Life Insurance Company, to whom the authors are deeply obliged.

Table of $T = \sum_{i=0}^{M^*-1} (M^* - i) \frac{e^{-N^*p}(N^*p)^i}{i!}$

$M^* - 1$	N^*p											
	1	2	3	4	5	6	7	8	9	10	11	12
1	1.10	.54	25	.11	.05	.02	.01	.00	.00	00	.00	00
2	2.02	1.22	67	.35	.17	.08	.04	.02	.01	.00	.00	.00
3	3.00	2.08	1 32	78	.44	.23	.12	.06	.03	.01	.01	.00
4	4 00	3.02	2.13	1.41	.88	.52	.29	.16	.08	.04	.02	.01
5	5.00	4.01	3.05	2.20	1.40	.96	.59	.35	.20	.11	.06	.03
6	6.00	5.00	4 02	3.08	2.26	1.57	1 04	.66	.41	.24	.14	.08
7	7.00	6.00	5.01	4.03	3.12	2.31	1.64	1.12	.73	.46	.28	.17
8	8 00	7.00	6.00	5.01	4 05	3 16	2.37	1.71	1.19	.79	.51	.32
9	9.00	8 00	7.00	6 00	5 02	4.08	3.20	2.43	1.77	1.25	.85	56
10	10 00	9 00	8.00	7.00	6 01	5.03	4 10	3.24	2.48	1.83	1.31	.91
11	11.00	10 00	9 00	8 00	7.00	6.01	5.05	4.13	3.28	2.53	1 89	1 37
12	12 00	11 00	10.00	9 00	8.00	7.01	6.02	5.07	4.16	3.32	2.58	1.95
13	13.00	12.00	11 00	10.00	9 00	8.00	7 01	6.03	5 08	4.19	3.36	2.63
14	14 00	13 00	12 00	11.00	10 00	9 00	8 00	7 01	6.04	5.10	4.22	3.40
15	15.00	14 00	13 00	12.00	11.00	10.00	9.00	8.01	7 02	6 05	5.12	4.25

8. The SP of H. F. Dodge. H. F. Dodge [1] has proposed a very interesting SP for continuous production. The plan is defined by two constants i and f and may be described as follows: Begin with complete inspection of the units consecutively as produced and continue such inspection until i units in succession are found non-defective. Thereafter inspect a fraction f of the units. Continue partial inspection until a defect is found. Then start complete inspection again and continue until i units in succession are found non-defective. Repeat the procedure.

Dodge [1] derived formulas for determining the AOQL corresponding to any

pair i and f, under the assumption that the production process is in a state of statistical control. Dodge's formulas for the AOQL are not necessarily valid if we do not make this restriction on the production process, i e., if we admit that the probability p that a unit will be defective may vary in any arbitrary way during the production process. This, of course, is not a criticism of the derivation of the formulas; it cannot be considered surprising that a formula is not valid under assumptions different from those under which it was derived. However, it is relevant to point out the fact that the Dodge SP does not guarantee the AOQL under all circumstances, so that care must be taken to ensure that certain requirements are met. Exactly what these requirements are is not known, statistical control is a sufficient condition, but is probably not necessary and could be weakened. It seems likely to the authors that, if p varies only slowly (with N) with infrequent "jumps," the Dodge SP will produce results which will exceed the AOQL by little, if at all. But if the "jumps" are numer-

Table of $T = \sum_{i=0}^{M^*-1} (M^* - i) \dfrac{e^{-N^*p}(N^*p)^i}{i!}$

(Continued)

$M^* - 1$	N^*p												
	13	14	15	16	17	18	19	20	21	22	23	24	25
1	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	00	.00	00
2	.00	.00	00	.00	.00	.00	.00	.00	.00	.00	00	.00	00
3	.00	00	.00	00	.00	.00	.00	.00	.00	00	00	.00	.00
4	.01	.00	.00	00	.00	.00	.00	.00	00	.00	.00	.00	00
5	.02	01	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	00
6	.04	.02	.01	.01	.00	.00	00	.00	.00	.00	00	.00	.00
7	.10	.05	.03	.02	.01	00	.00	.00	.00	.00	.00	.00	00
8	.20	12	07	.04	.02	01	.01	.00	.00	00	.00	.00	00
9	.36	.23	14	08	.05	.03	.01	01	.00	.00	.00	00	00
10	.61	40	.26	.16	.10	.06	.03	.02	.01	.01	.00	.00	.00
11	.97	66	44	29	.18	.11	.07	.04	.02	01	.01	.00	00
12	1.43	1.02	.71	.48	.32	20	.13	08	05	.03	02	.01	.01
13	2 00	1.48	1.07	.75	.52	.35	.23	.15	.09	.06	.03	.02	01
14	2 68	2 05	1.54	1.12	.80	.55	.38	.25	16	.10	.07	.04	.02
15	3 44	2.72	2.10	1.59	1.17	84	.59	.11	.27	.18	.12	.07	.05

ous and appropriately spaced it is possible to exceed the AOQL by substantial amounts, as the example below will show. The Dodge plan was intended to serve as an aid to the detection and correction of malfunctioning of the production process and this use would tend to prevent the occurrence of such a phenomenon. Parenthetically, it should be remarked that the information obtained in the course of inspection according to either the plans discussed in this paper or any reasonable scheme should, if possible, be sent at once to the producing divisions for their guidance.

An example to show that the AOQL can be exceeded can be constructed as

follows: Let $i = 54$ and $f = 0.1$. Then according to the graphs of [1], page 272, the AOQL should be 0.02. Define a sequence of 60 successive units free of defectives as a segment of type 1, and a sequence of 60 successive units where the production process is in statistical control with $p = 0.1$, as a segment of type 2. Suppose that the sequence of units produced consists of segments of types 1 and 2 always alternating. Then it follows that the first item inspected in a segment of type 2 is always inspected on a partial inspection basis. We now assume that, unless the occurrence of a defective has previously terminated partial inspection, the 1st, 11th, 21st, 31st, 41st, and 51st items in a segment of type 2 will be chosen for partial inspection, and if the 1st item is found defective, the entire segment of type 2 will be cleared of defectives. (Both of these assumptions favor the Dodge SP.) Then the situation is as described in the following table:

	(1) *Probability of first terminating partial inspection at each item*	(2) *Expected number of defectives remaining in segment of type 2 after partial inspection has been terminated*	(3) (1) x (2)
1st	.1	0	0
11th	$(.9)\,(.1) = .09$	.9	.081
21st	$(.9)^2(.1) = .081$	1.8	.1458
31st	$(.9)^3(.1) = .0729$	2.7	.19683
41st	$(.9)^4(.1) = .06561$	3.6	.236196
51st	$(.9)^5(.1) = .059049$	4.5	.2657205

Probability that an entire segment of type 2 will be partially inspected	*Expected number of defectives left in a segment of type 2 which has been inspected only partially*	*Product*
$(.9)^6 = .531441$	5.4	2.8697814
		Sum = 3.7953279

The AOQ is therefore $\frac{3.7953279}{120} = .0316+$, while $L = .02$.

It is therefore difficult to compare the Dodge plan with any of the plans described in this paper with respect to their effect on a production process not in statistical control. If the production process is in statistical control, then, as we have already seen, SPA requires minimum inspection (and, incidentally, because of the existence of statistical control, produces a fair degree of local stability). If, when statistical control exists, one requires both maintenance of a given AOQL and a higher degree of local stability than is produced by SPA, the relevant comparison is between the Dodge plan and SPC. Both will probably give good results as regards local stability, but it is not possible at present to make

these intuitive notions precise, as we have not given an exact definition of local stability. The following example (in which statistical control is assumed) may not be unrepresentative of what the situation is with regard to the amount of inspection required.

Fraction of product inspected under the Dodge plan and under SPC when
$L = .045$ $f = .1$

p	Fraction of product inspected under the Dodge plan	Fraction of product inspected under SPC when		
		$N_0 = 400$	$N_0 = 1000$	$N_0 = 2000$
.01	.12	.12	.10	.10
.02	.15	.17	.11	.10
.03	.19	.22	.14	.11
.04	.23	.28	.19	.15
.05	.28	.34	.26	.21
.06	.33	.40	.33	.29
.07	.39	.45	.39	.37
.08	.45	50	.46	.44
.09	.52	.54	.51	.50
.10	.58	.57	.55	.55

The decrease in inspection required by SPC as N_0 increases is evident in this table. When $N_0 = 2000$ SPC requires less inspection than the Dodge plan, when $N_0 = 400$ it requires more inspection than the Dodge plan. How the various degrees of local stability achieved compare remains an open question. The case when $N_0 = 400$ probably lies in the region where SPC is inefficient (as regards amount of inspection) and corresponds to a high degree of local stability.

We note that both plans call for increased inspection as the quality worsens (p increases). If the manufacturer is required to pay for the inspection this serves as an added incentive to improve quality of output.

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THE EXPECTED VALUE AND VARIANCE OF THE RECIPROCAL AND OTHER NEGATIVE POWERS OF A POSITIVE BERNOULLIAN VARIATE[1]

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1. Introduction. The expected value of the reciprocal of a Bernoullian variate appears in certain problems of random sampling wherein both practical considerations and mathematical necessity make zero an inadmissible value of the variate. This special condition excluding zero is necessary from a practical standpoint because statistics can not be calculated from an empty class. It is a necessary condition, in the mathematical sense, for the expected value, and variances involving it, to be finite. When subject to this condition the Bernoullian variate will be designated the *positive* Bernoullian variate.

There appears to be no simple expression for the expected value of the reciprocal such as there is for the expected value of positive integral powers of the positive Bernoullian variate. This paper presents in (15) a factorial series, which can be computed conveniently to any desired number of terms by means of the recursion relation (18). Upper and lower bounds on the remainder may be computed readily from (20), (21), (23), (24), and (26) and the approximation may be improved by adding an estimate of the remainder taken between these bounds. A factorial series for the expected value of negative integral powers is given in (34). A factorial series for the expected value of the reciprocal of the positive hypergeometric variate is given in (53). Series for the variances follow directly from the series for expected values.

A simple example of the sampling problems in which this expected value appears is presented by the following instance of estimates derived from samples of variable size:

An infinite population consists of items of two kinds or classes, A and B. Lots of N items each are drawn at random. In such lots the number of items, x', that are of class A is an ordinary Bernoullian variate. Next, every lot composed entirely of items of class B is discarded. This excludes all lots for which $x' = 0$. From each remaining lot the $N - x'$ items of class B are set aside, leaving a sample composed entirely of items of class A. The number of such items, x, varies from sample to sample. It will be designated a positive Bernoullian variate since $x = x'$ if $x' > 0$ and x does not exist if $x' \leq 0$. Finally, let there be associated with each item in class A a particular value of a variable, y, the variance of which in A is σ^2. Then if the mean value of y is computed for each sample, the error variance of such means is $E(\sigma^2/x) = \sigma^2 E(1/x)$.

Instances similar to that just described occur in the design of sampling surveys from which statistics are to be obtained separately for each of several classes

[1] Developed from a section of a paper presented to the Washington meeting of the Institute of Mathematical Statistics on June 18, 1943.

of the population, i.e., each statistic is to be computed from some part of the sample instead of all of it. They also occur in certain sampling problems in which some of the items drawn for a sample turn out to be blanks

A related problem concerning the error variance of the proportion of males among infants born in any one year was considered by G. Bohlmann in a paper on approximations to the expected value and standard error of a function [1]. His approach to the problem was to expand the function in a Taylor series and take the expected value of each term. The conditions under which the resulting series converges were developed for certain functions of a Bernoullian variate. The present paper provides a different and, in certain respects, superior approach to the problem employing a method due to Stirling [2]. While the method is applied to the reciprocal and negative powers it is also applicable to certain other functions of a Bernoullian variate.

2. The positive Bernoullian variate. Let x be a random variate defined by a Bernoullian probability function subject to the special condition $x > 0$. The probability of x in n is

$$P(x) = \binom{n}{x} p^x q^{n-x}/(1 - q^n) \tag{1}$$

where x and n are integers, $1 \leq x \leq n$, and

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}. \tag{2}$$

The probabilities p and q are constants, $0 < p = 1 - q < 1$.

The divisor $1 - q^n$ arises from the condition excluding zero. (Bohlmann omits this factor, assuming that q^n is negligible, an assumption that is not always valid. In fact, $q^n \sim e^{-np}$.) An extension of this condition to exclude all values of x less than a specified constant will be considered in a later section.

Throughout this paper summation is understood to be from $x = 1$ to $x = n$ unless it is shown otherwise.

3. Expected values and moments. The expected values of x and its positive integral powers are

$$E(x) = np/(1 - q^n) \tag{3}$$

$$E(x^2) = (npq + n^2p^2)/(1 - q^n) \tag{4}$$

and, in general

$$E(x^i) = \nu_i/(1 - q^n) = \sum_j \mathfrak{S}_i^j j! \binom{n}{j} \frac{p^j}{1 - q^n}, \qquad i > 0 \tag{5}$$

where ν_i is the ith moment about zero of an ordinary Bernoullian variate with the same n and p and the $\mathfrak{S}_i^j$ are the Stirling numbers of the second kind (see Table 1).

The moments about $E(x)$ are somewhat more complicated than the corre-

sponding moments of the ordinary Bernoullian variate. For example, the variance

$$E\{(x - E(x))^2\} = \frac{npq}{1 - q^n} - \frac{n^2p^2q^n}{(1 - q^n)^2} \tag{6}$$

and the third moment

$$E\{(x - E(x))^3\} = \frac{(q - p)npq}{1 - q^n} - \frac{3n^2p^2q^{n+1}}{(1 - q^n)^2} + \frac{n^3p^3q^n(1 + q^n)}{(1 - q^n)^3}. \tag{7}$$

The moments about np, the first moment of an ordinary Bernoullian variate, are

$$E\{(x - np)^i\} = (\mu_i + (-1)^{i-1}(np)^i q^n)/(1 - q^n) \tag{8}$$

TABLE 1

Stirling numbers of the second kind, $\mathfrak{S}_i^j$

$i \backslash j$	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	3	1	0	0	0
4	1	7	6	1	0	0
5	1	15	25	10	1	0
6	1	31	90	65	15	1
7	1	63	301	350	140	21
8	1	127	966	1,709	1,050	266
9	1	255	3,025	7,770	6,951	2,646
10	1	511	9,330	34,105	42,525	22,827

where μ_i is the ith moment, about the mean, of an ordinary Bernoullian variate with the same values n and p.

The expected value of the reciprocal is

$$E\left(\frac{1}{x}\right) = \frac{1}{1 - q^n}\left\{\frac{1}{1}npq^{n-1} + \frac{1}{2}\cdot\frac{1}{2}n(n - 1)p^2q^{n-2} + \cdots + \frac{1}{i}\binom{n}{i}p^iq^{n-i} + \cdots + \frac{1}{n}p^n\right\}. \tag{9}$$

This equation is not suitable for the computation of $E(1/x)$ to a satisfactory degree of approximation unless np is small, say less than 5 for most purposes. The number of terms necessary to obtain a computed value with four significant figures, for example, may be estimated to be approximately $8\sqrt{npq}/(- q^n)$.

Expressed as a function of q, $E(1/x)$ becomes

$$E\left(\frac{1}{x}\right) = \frac{1}{1 - q^n}\Sigma\frac{q^{x-1} - q^n}{n - x + 1} \tag{10}$$

a series which may be convenient for small values of q.

$E(1/x)$ may be expanded in a power series by Taylor's Theorem. It may

also be expanded in a finite series of expected values of powers, either in $E(x)$, $E(x^2)$, $\cdots$ or in $E(x - c)$, $E(x - c)^2$, $\cdots$ c being any positive constant. The second of these three series may be obtained by expanding $\frac{1}{x}\left(1 - \frac{x}{c}\right)^t$ and taking expected values, and the third by dividing out $\frac{1}{x} = \frac{1}{c + (x - c)}$ and taking expected values. For all three expansions, however, the terms become progressively more complicated and laborious to compute. A simpler and more convenient series for actual computations may be obtained by expanding $1/x$ in a factorial series.

4. Expansion of $E(1/x)$ in a series of inverse factorials. It is easy to prove by induction that, $x > 0$,

$$\frac{1}{x} = \frac{0!}{x+1} + \frac{1!}{(x+1)(x+2)} + \cdots + \frac{(i-1)!x!}{(x+i)!} + \cdots + \frac{(t-1)!x!}{(x+t)!} + R_t(x) \tag{11}$$

where

$$R_t(x) = t!(x-1)!/(x+t)! \tag{12}$$

is the remainder after the first t terms. This is, of course, an expansion in Beta functions. It is also a simple special case of the expansion of a function in a "faculty series" or series of inverse factorials [3] with an exact expression for the remainder.

Let

$$s_i = \Sigma \binom{n+i}{x+i} \frac{p^{x+i} q^{n-x}}{1 - q^n} = \frac{1}{1 - q^n}\left(1 - \sum_{x=0}^{i} \binom{n+i}{x} p^x q^{n+i-x}\right). \tag{13}$$

Then, since

$$\Sigma \frac{x!}{(x+1)!} \binom{n}{x} p^x q^{n-x} = \frac{n!\, s_i (1 - q^n)}{(n+i)!\, p^i} \tag{14}$$

the expected value of (11) is

$$E\left(\frac{1}{x}\right) = \frac{0!\, s_1}{(n+1)p} + \frac{1!\, s_2}{(n+1)(n+2)p^2} + \cdots + \frac{(i-1)!\, n!\, s_i}{(n+1)!\, p^i} + \cdots + \frac{(t-1)!\, n!\, s_t}{(n+t)!\, p^t} + \Sigma R_t(x) P(x). \tag{15}$$

When developed as infinite series, both (11) and (15) are convergent since the remainders $R_t(x) \to 0$ as $t \to \infty$.

For computing purposes it is convenient to write

$$E\left(\frac{1}{x}\right) = \sum_{i=1}^{t} u_i + E(R_t(x)) \tag{16}$$

in which, since

$$s_i = s_{i-1} - q\binom{n+i-1}{i} \frac{p^i q^{n-1}}{1 - q^n}, \tag{17}$$

the following recursion relation exists between u_i and u_{i-1}

$$u_i = \frac{(i-1)!\,n!\,s_i}{(n+i)!\,p^i} = \frac{(i-1)u_{i-1} - k/i}{(n+i)p}, \qquad i > 1; \tag{18}$$

$$u_1 = \frac{1-k}{(n+1)p}$$

where

$$k = npq^n/(1-q^n) \sim np/(e^{np}-1). \tag{19}$$

This reduces the computing of the u_i to a simple repetitive procedure. The computing is still simpler in those problems in which, for the degree of precision desired, k is negligible.

An estimate of $E(R_t(x))$ should be added to the sum in (16) to improve the approximation. To determine a suitable estimate, a lower bound for the expected value of the remainders may be computed from one of the following inequalities:

$$\begin{aligned} E(R_t(x)) &= \Sigma \frac{t}{x}\,\frac{(t-1)!\,x!}{(x+t)!}\,P(x) \\ &= \Sigma t\left(\frac{1}{m} - \frac{x-m}{m^2} + \frac{(x-m)^2}{m^2x}\right)\frac{(t-1)!\,x!}{(x+t)!}\,P(x) \\ &> \frac{1}{m}\,tu_t - \frac{1}{m^2}\,t(t-1)u_{t-1} + \frac{m+t}{m^2}\,tu_t, \qquad m \neq 0 \end{aligned} \tag{20}$$

which is maximized by setting $m = \{(t-1)u_{t-1} - tu_t\}/u_t$, whence

$$E(R_t(x)) > tu_t^2/\{(t-1)u_{t-1} - tu_t\}, \qquad t > 1. \tag{21}$$

Also, since when $m = E(x)$

$$\Sigma(x-m)\,\frac{(t-1)!\,x!}{(x+t)!}\,P(x) < \Sigma(x-m)P(x) = 0, \tag{22}$$

a simpler inequality is

$$E(R_t(x)) > tu_t(1-q^n)/np. \tag{23}$$

Further, if only the first $c < n$ terms in (20) are taken,

$$E(R_t(x)) > \sum_{x=1}^{c} \frac{t!\,(x-1)!}{(x+t)!}\,P(x) = \sum_{x=1}^{c} v_x \tag{24}$$

where

$$v_1 = \frac{k}{(t+1)q} \quad \text{and} \quad v_x = \frac{(x-1)(n-x+1)p}{x(x+t)q}\,v_{x-1}. \tag{25}$$

An upper bound may be computed from

$$E(R_t(x)) < \begin{cases} tu_t & (26.1) \\[4pt] \frac{1}{2}\,tu_t + \frac{1}{2}\,v_1 & (26.2) \\[4pt] \frac{1}{3}\,tu_t + \frac{2}{3}\,v_1 + \frac{1}{6}\,v_2 & (26.3) \\ \quad\cdots & \\ \frac{1}{j}\,tu_t + \sum_{x=1}^{j-1}\left(\frac{1}{x} - \frac{1}{j}\right)v_x & (26.j) \end{cases} \tag{26}$$

the choice among which may be governed by computing convenience. Taken with (16), these inequalities provide lower and upper bounds for $E(1/x)$.

5. Examples. Two examples will serve to illustrate the factorial series (15).

EXAMPLE 1

Computation of $E(1/x)$ for $n = 100$ and $p = 0.1$

$np = 10 \qquad k = .000,265,621 \qquad E(1) = .111,527$

t	*Binomial sum of t terms*	*Sum of t terms*	*Factorial series lower bounds**		*Upper bound***
1	.000,295	.098,984	.099,647		.132,167
2	.001,107	.108,675	.109,006	(.111,034)	.115,247
3	.003,071	.110,548	.110,752	(.111,313)	.112,498
4	.007,039	.111,082	.111,223	(.111,381)	.111,852
5	.013,813	.111,280	.111,385	(.111,452)	.111,657
6	.023,743	.111,370	.111,452	(.111,478)	.111,587
7	.036,442	.111,416	.111,483	(111,489)	.111,556
8	.050,796	.111,444	.111,500	(.111,497)	.111,544
9	.065,287	.111,461	.111,509	(111,503)	.111,537
10	.078,474	.111,472	.111,514	(.111,508)	111,534
11	.089,372	.111,481	.111,518	(.111,511)	.111,532
12	.097,604	.111,487	.111,520		.111,530
13	.103,320	.111,492	.111,521		111,529
14	.106,985	.111,495	.111,523		.111,529
15	.109,164	.111,498	.111,524		.111,529
16	.110,369	.111,501	.111,524		.111,528
17	.110,992	.111,503	.111,525		.111,528
18	.111,294	.111,505	.111,525,4		.111,527,5
19	.111,431	.111,506	.111,525,6		.111,527,3
20	.111,489	.111,508	.111,525,8		.111,527,1
..					
24	.111,526				
...					
100	.111,527 (end of series)				

* Sum of t terms plus lower bound for $E(R(x))$ from (24) with $c = 3$. Numbers in parentheses are calculated from (21).

** Sum of t terms plus upper bound on $E(R(x))$ from (26.3).

EXAMPLE 2

Computation of $E(1/x)$ for $n = 1000$ and $p = 0.3$

$np = 300 \qquad k = 9.7 \times 10^{-14}$

t	*Sum of t terms*	*Factorial series upper and lower bounds**
1	.003,330,003,330	
2	.003,341,081,185	.003,346,7 .003,341,0 (.003,341,155,4)

* Computed as in Example 1.

t	*Sum of t terms*	*Factorial series upper and lower bounds**
3	.003,341,154,817	$\begin{cases} .003,341,211 \\ .003,341,155 \end{cases}$
4	.003,341,155,549	$\begin{cases} .003,341,156,29 \\ .003,341,155,56 \end{cases}$
5	.003,341,155,559	$\begin{cases} .003,341,155,58 \\ .003,341,155,57 \end{cases}$

For the binomial series, the sum of the largest eight terms of (9), not the first eight terms, is approximately .0007 which is less than 1/4 of the value of $E(1/x)$.

In the first example the value of np is almost small enough to make computation by (9) convenient. In the second example about 120 terms of (9) must be computed to obtain an approximation to four significant figures but only four terms of the factorial series are needed to obtain seven significant figures. It is evident that as np increases, the number of terms of (16) required to obtain an approximation to a given number of significant figures decreases. The opposite is true of (9) as n increases, or as p approaches a value near 1/2.

6. Extending the special condition. In some sampling problems all values of x less than a specified value, g, and greater than another specified value, h, are inadmissible. Then the probability of x in n is

$$P(x \mid g, h) = \binom{n}{x} p^x q^{n-x} / s_{0,g,h}, \qquad g \leq x \leq h, \tag{27}$$

where

$$s_{0,g,h} = \sum_{x=g}^{h} \binom{n}{x} p^x q^{n-x}. \tag{28}$$

With this new condition, $E(1/x)$ is given by (15) if s_i is replaced by

$$s_{i,g,h} = \sum_{x=g}^{h} \binom{n+i}{x+i} \frac{p^{x+i} q^{n-x}}{s_{0,g,h}} \tag{29}$$

and the summation in the remainder term is from g to h. Also since

$$s_{i,g,h} = s_{i-1,g,h} - \frac{q}{s_{0,g,h}} \left\{ \binom{n+i-1}{g+i-1} p^{g+i-1} q^{n-g} - \binom{n+i-1}{h+i} p^{h+i} q^{n-h-1} \right\} \tag{30}$$

a recursion relation similar to (18) may be used in computing

$$
\begin{aligned}
u_{i,g,h} &= \frac{(i-1)!\,n!\,s_{i,g,h}}{(n+i)!\,p^i} \\
&= \frac{(i-1)u_{i-1,g,h} - (i-1)!\,\{k_g/(g+i-1)! + k_{h+1}/(h+i)!\}}{(n+i)p}
\end{aligned}
\tag{31}
$$

where

$$k_g = \frac{n!\,p^g q^{n-g+1}}{(n-g)!\,s_{0,g,h}} \tag{32}$$

$$k_h = \frac{n!\,p^h q^{n-h+1}}{(n-h)!\,s_{0,g,h}}. \tag{33}$$

The inequalities (20) to (23) inclusive and (26) are applicable to this extension on substitution of $u_{i,g,h}$ for u_i.

7. Expansion of $E(x^{-a})$ in a factorial series. Equation (11) may be extended to other negative integral powers of x. If a is a positive integer

$$
\begin{aligned}
E(x^{-a}) = \Sigma \frac{1}{x^a} P(x) &= \frac{b_{1,a} s_1}{(n+1)p} + \frac{b_{2,a} s_2}{(n+1)(n+2)p^2} \\
&\quad + \cdots + \frac{b_{t,a} s_t\, n!}{(n+t)!\,p^t} + \Sigma R'_t(x) P(x)
\end{aligned}
\tag{34}
$$

where

$$R'_t(x) = \sum_{j=1}^{a} b_{t+1,j} \frac{x^{j-1}\,x!\,P(x)}{(x+t)!\,x^a} \tag{35}$$

and the $b_{i,j}$ are the absolute values of the Stirling numbers of the first kind (see Table 2) formed by the recursion relation

$$b_{i,j} = b_{i-1,j-1} + (i-1)b_{i-1,j}, \qquad b_{i,j} = 0 \quad \text{if} \quad j > i \quad \text{or} \quad j < 1. \tag{36}$$

It is evident that

$$\sum_{j=1}^{i} b_{i,j} = i! \tag{37}$$

$$b_{i,1} = (i-1)! \quad \text{and} \quad b_{i,j} < i! \quad \text{if} \quad j > 1, \tag{38}$$

whence

$$
\begin{aligned}
R'_t(1) &= \frac{1}{t+1} P(1) \\
R'_t(x) &< \frac{((t+1)! - t!)\,x!\,P(x)}{2(x+t)!}, \qquad x > 1 \\
&< \frac{1}{(t+1)} P(x).
\end{aligned}
\tag{39}
$$

Hence $R_t'(x) \to 0$ and $E(R_t'(x)) \to 0$ as $t \to \infty$ and the sum of the first t terms of (34) converges to $E(x^{-a})$ as $t \to \infty$.

The following recursion relation corresponding to (18) provides a simple procedure for computing:

$$u_{i,a} = b_{i,a} u_i/(i-1)! = b_{i,a} \frac{(u_{i-1,a}/b_{i-1,a}) - k/i!}{(n+1)p}. \tag{40}$$

The computing procedure, then, follows a cycle of four simple operations:
1. Divide $\{k/(i-1)!\}$ by i.
2. Subtract the quotient from $\{u_{i-1,a}/b_{i-1,a}\}$.
3. Divide the difference by $\{(n+i+1)p\} + p$. The quotient is $u_{i,a}/b_{i,a}$.
4. Multiply this quotient by $b_{i,a}$.

TABLE 2

*Absolute values of Stirling numbers of the first kind, $b_{i,j}$**

i \ j	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	2	3	1	0	0	0
4	6	11	6	1	0	0
5	24	50	35	10	1	0
6	120	274	225	85	15	1
7	720	1,764	1,624	735	175	21
8	5,040	13,068	13,132	6,769	1,960	322
9	40,320	109,584	118,124	67,284	22,449	4,536
10	362,880	1,026,576	1,172,700	723,680	269,325	63,273

* These numbers are also known as differential coefficients of zero [4].

The expressions in braces are quantities obtained in the preceding cycle.

The $u_{i,a}$ may also be calculated from (18), or checked by such a calculation.

A lower bound for $E(R'(x))$ after t terms may be calculated from the first c terms of

$$\begin{aligned} E(R'(x)) &= \sum R_t'(x)P(x) > \sum_{x=1}^{c} R_t'(x)P(x) \\ &= \sum_{x=1}^{c}\sum_{j=1}^{a} \frac{b_{t+1,j}\, n!\, p^x q^{n-x}}{x^{a-j+1}(x+t)!\,(n-x)!\,(1-q^n)} \end{aligned} \tag{41}$$

or from an inequality similar to (23)

$$E(R'(x)) > \frac{u_t}{(t-1)!} \sum_{j=1}^{a} \frac{b_{t+1,j}}{(E(x))^{a-j+1}} \tag{42}$$

which may also be written

$$E(R'(x)) > \frac{u_t}{(t-1)!(E(x))^{a+1}}\Big\{(E(x)+t)(E(x)+t-1)\cdots E(x) \tag{43}$$
$$- \sum_{j=a+1}^{t+1} b_{t+1,j}(E(x))^j\Big\}.$$

An upper bound may be calculated from

$$E(R'(x)) < \frac{u_t}{(t-1)!}\sum_{j=1}^{a} b_{t+1,j} < t(t+1)u_t \tag{44}$$

or

$$E(R'(x)) < \sum_{x=1}^{c} R'(x)P(x) + \sum_{x=c+1}^{n}\sum_{j=1}^{a} b_{t+1,j}\frac{x!P(x)}{(x+t)!c^{a-j+1}}$$
$$< \sum_{x=1}^{c} R'(x)P(x) + \frac{u_t}{(t-1)!}\sum_{j=1}^{a}\frac{b_{t+1,j}}{c^{a-j+1}} = \sum_{x=1}^{c} R'(x)P(x) \tag{45}$$
$$+ \frac{u_t}{(t-1)!c^{a+1}}\Big\{(c+t)(c+t-1)\cdots c - \sum_{j=a+1}^{t+1} b_{t+1,j}c^j\Big\}.$$

8. The positive hypergeometric variate. The theory of sampling without replacement from a finite population rests on the hypergeometric variate. Its probability function is

$$P(x \mid N, M, n) = \binom{M}{x}\binom{N-M}{n-x}\Big/\binom{N}{n}. \tag{46}$$

In applications to finite sampling, N is the number of items in the population, M is the number of them that are of a certain kind, n is the number of items drawn for the sample, and x is the number of items of the designated kind in the sample.

As in the case of the Bernoullian variate, it is necessary to exclude zero in defining the expected value of $1/x$. The probability function of the positive hypergeometric variate, then, is

$$P_H(x) = P(x \mid N, M, n)/s_0, \qquad x > 0 \tag{47}$$

where

$$s_0 = 1 - P(0 \mid N, M, n). \tag{48}$$

Throughout this section the notation will have reference to (47) instead of (1). The expected values of positive integral powers of x are

$$E(x) = Mn/(Ns_0) \tag{49}$$

$$E(x^2) = \frac{1}{s_0}\left\{\frac{M(M-1)n(n-1)}{N(N-1)} + \frac{Mn}{N}\right\} \tag{50}$$

and, in general,

$$E(x^i) = \sum_{j=1}^{i} \mathfrak{S}_i^j E(x!/(x-j)!) \tag{51}$$

where the $\mathfrak{S}_i^j$ are the Stirling numbers of the second kind and

$$E\left(\frac{x!}{(x-j)!}\right) = \frac{M!n!(N-j)!}{(M-j)!(n-j)!N!s_0}. \tag{52}$$

The factorial series corresponding to (16) is

$$E\left(\frac{1}{x}\right) = \sum \frac{1}{x} P_H(x) = \sum_{i=1}^{t} u_i + E(R_t(x)) \tag{53}$$

where

$$u_i = \Sigma \frac{(i-1)!x!}{(x+i)!} P_H(x) \tag{54}$$

and

$$E(R_t(x)) = \Sigma \frac{t!(x-1)!}{(x+t)!} P_H(x). \tag{55}$$

The u_i may be computed from

$$\begin{aligned} u_1 &= \frac{(N+1)s_1}{(M+1)(n+1)s_0} \\ &= \frac{1}{s_0}\left\{\frac{N+1}{(M+1)(n+1)} - \frac{(N-M)!(N-n)!}{N!(N-M-n-1)!(M+1)(n+1)}\right\} \end{aligned} \tag{56}$$

and the recursion relation

$$u_i = \frac{(N+i)s_i}{(M+i)(n+i)s_{i-1}} u_{i-1} \tag{57}$$

where

$$s_i = 1 - \sum_{x=0}^{i} P(x \mid N+i, M+i, n+i). \tag{58}$$

The computing is quite simple in those instances in which $1 - s_i$ is negligible.

Corresponding to (26), an upper bound for the expected value of the remainders after t terms may be computed from

$$E(R_t(x)) < \begin{cases} tu_t & (59.1) \\ \frac{1}{2}tu_t + \frac{1}{2}P_H(1)/(t+1) & (59.2) \\ \frac{1}{3}tu_t + \frac{2}{3}\frac{P_H(1)}{t+1} + \frac{1}{6}\frac{P_H(2)}{(t+1)(t+2)} & (59.3) \\ \cdots & \\ \frac{1}{j}tu_t + t!\sum_{x=1}^{j-1}\left(\frac{1}{x} - \frac{1}{j}\right) P_H(x)\frac{x!}{(x+t)!}. & (59.j) \end{cases} \tag{59}$$

A lower bound for the expected value of the remainders may be computed from one of the following inequalities corresponding to (23), (21) and (24)

$$E(R_t(x)) > tu_t N s_0/(Mn) \tag{60}$$

$$E(R_t(x)) > tu_t^2/\{(t-1)u_{t-1} - tu_t\} \tag{61}$$

$$E(R_t(x)) > \sum_{x=1}^{t} \frac{t!\,(x-1)!}{(x+t)!} P_H(x). \tag{62}$$

The expected values of other negative integral powers of the positive hypergeometric variate may be calculated from

$$E(x^{-a}) = \sum_{i=1}^{t} b_{i,a} u_i/(i-1)! + E(R_t'(x)) \tag{63}$$

where

$$R_t'(x) = \sum_{j=1}^{a} b_{t+1,j} \frac{x^{j-1} x!\, P_H(x)}{x^a (x+t)!}. \tag{64}$$

With $P_H(x)$ substituted for $P(x)$, (39), (42), (43), (44), and (45) provide lower and upper bounds for $E(R_t'(x))$ for the positive hypergeometric variate. Also, corresponding to (41)

$$E(R_t'(x)) > \sum_{x=1}^{c} R_t'(x) P_H(x). \tag{65}$$

9. Variance and moments of $1/x$ and x^{-a}. The variance of $1/x$, which is $E(1/x^2) - (E(1/x))^2$, may be calculated from (16) and (34), with $a = 2$, for the positive Bernoullian variate, and from (53) and (63), with $a = 2$, for the positive hypergeometric variate. Likewise, the variance of x^{-a} and the moments of $1/x$ and x^{-a} about $E(1/x)$ may be computed by the usual formulae.

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[3] The writer is indebted to Dr. Felix Bernstein for the reference to Bohlman.

RANDOM WALK IN THE PRESENCE OF ABSORBING BARRIERS

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1. Introduction. The problem of random walk (along a straight line) in the presence of absorbing barriers can be stated as follows:

A particle, starting at the origin, moves in such a way that its displacements in consecutive time intervals, each of duration Δt, can be represented by independent random variables

$$X_1, X_2, X_3, \cdots$$

Moreover, if at some time the total (cumulative) displacement becomes $>p$ ($p \geq 0$) or $< -q$ ($q \geq 0$) the particle gets absorbed. The problem is to determine the probability that "the length of life" of the particle is greater than a given number t. This problem also admits an interpretation in terms of a game of chance in which the player quits when he loses more than q or wins more than p. An interesting paper on this type of problem by A. Wald[1] appeared recently in the *Annals*. Wald assumes that the X's are identically distributed and that their mean and standard deviation are different from 0.[2] He is then mostly interested in the limiting case when both the mean and the standard deviation become small. The object of this paper is to propose a different method of attack which in some cases leads to an answer in closed form. The method we use has been employed repeatedly in statistical mechanics in the study of the so called order-disorder problem. It is due, I believe, to E. W. Montroll[3]. As far as the author knows this method was never used in connection with the classical probability theory and this seems to furnish an additional reason for publishing this paper.

2. The simplest discrete case. We assume that each X is capable of assuming the values 1 and -1 each with probability $\frac{1}{2}$, and for simplicity sake we let $\Delta t = 1$. Note that, unlike in Wald's case, the mean of X is 0. Denote by N the random variable which represents the "length of life" of the particle and let (m an integer)

$$\delta(m) = \begin{matrix} \frac{1}{2} & m = 1 \quad \text{or} \quad m = -1, \\ 0 & \text{otherwise.} \end{matrix}$$

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[1] A. Wald "On cumulative sums of random variables," *Annals of Math. Stat.*, Vol. 15 (1944), pp. 283–296.

[2] Since this was written Professor Wald informed the author that he can easily avoid the condition that the mean should be zero.

[3] See for instance E W. Montroll, "Statistical Mechanics of nearest neighbor systems," *Jour. of Chem. Physics*, Vol. 9 (1941), pp 706–721.

Clearly we have (throughout this section we assume that both p and q are integers)

$$\text{Prob. } \{N > n\} = \text{Prob. } \{-q \leq X_1 \leq p, -q \leq X_1 + X_2 \leq p, \cdots, -q \leq X_1 + \cdots + X_b \leq p\} = \Sigma\delta(m_1)\delta(m_2) \cdots \delta(m_n),$$

where the summation is extended over all integers $m_1, m_2, \cdots m_n$ for which

$$-q \leq m_1 \leq p, \quad -q \leq m_1 + m_2 \leq p, \cdots, \quad -q \leq m_1 + m_2 + \cdots + m_n \leq p.$$

Letting

$$l_j = q + m_1 + \cdots + m_j, \qquad (j = 1, 2, \cdots, n),$$

we see that

$$\text{(1)} \qquad \text{Prob } \{N > n\} = \sum_{l_1, \cdots, l_n = 0}^{p+q} \delta(l_1 - q)\delta(l_2 - l_1) \cdots \delta(l_n - l_{n-1}).$$

Let us now consider the $(p + q + 1)$ by $(p + q + 1)$ matrix

$$\text{(2)} \qquad A = ((\delta(i - k))) = \begin{Vmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{Vmatrix}.$$

It is easily seen that the sum in (1) is equal to the sum of the elements in the $(q + 1)$-st column (or row) of the matrix A^n. Thus

Prob. $\{N > n\}$ = sum of the elements of the $(q + 1)$-st column of A^n.

Denote by $\lambda_1, \lambda_2, \cdots \lambda_{p+q+1}$ the eigenvalues of the matrix A and let

$$(x_1^{(j)}, x_2^{(j)}, \cdots, x_{p+q+1}^{(j)})$$

be the normalized eigenvector of A belonging to the eigenvalue λ_j. It can be shown by elementary means[4] that

$$\lambda_j = \cos \frac{\pi j}{p + q + 2}$$

[4] Matrices of type (2) have been introduced and studied in various connections. In a paper by R. P. Boas and the present author recently accepted by the *Duke Mathematical Journal* references to several authors are given. In order to find the eigenvalues and the eigenvectors of (2) it suffices to know that

$$\begin{vmatrix} 1 & a & 0 & \cdots & \cdots \\ a & 1 & a & \cdots & \cdots \\ 0 & a & 1 & a & \cdots \\ 0 & 0 & a & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \frac{\rho_1^{m+1} - \rho_2^{m+1}}{\rho_1 - \rho_2},$$

where m is the order of the matrix ρ_1 and ρ_2 roots of the equation $\rho^2 - \rho + a^2 = 0$.

and

$$x_k^{(j)} = \frac{\sqrt{2}}{\sqrt{p+q+2}} \sin \frac{\pi jk}{p+q+2}.$$

Denoting by R the orthogonal matrix

$$\begin{pmatrix} x_1^{(1)} & x_2^{(1)} & \cdots & x_{p+q+1}^{(1)} \\ x_1^{(2)} & x_2^{(2)} & \cdots & x_{p+q+1}^{(2)} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{(p+q+1)} & x_2^{(p+q+1)} & \cdots & x_{p+q+1}^{(p+q+1)} \end{pmatrix}$$

and by R' the transposed of R we have (since the eigenvalues of A are simple) by a well known theorem

$$A^n = R' \begin{pmatrix} \lambda_1^n & & & 0 \\ & \lambda_2^n & & \\ & & \ddots & \\ 0 & & & \lambda_{p+q+1}^n \end{pmatrix} R.$$

It thus follows by an easy computation that the sum of the elements of the $(q+1)$-st column (row) of A^n is

$$\sum_{r=1}^{p+q+1} \sum_{j=1}^{p+q+1} \lambda_j^n x_r^{(j)} x_{q+1}^{(j)} = \sum_{j=1}^{p+q+1} \lambda_j^n x_{q+1}^{(j)} \left(\sum_{r=1}^{p+q+1} x_r^{(j)} \right).$$

We have

$$\sum_{r=1}^{p+q+1} x_r^{(j)} = \frac{\sqrt{2}}{\sqrt{p+q+2}} \sum_{r=1}^{p+q+1} \sin \frac{\pi jr}{p+q+2}$$

$$= \begin{cases} 0, & j \text{ even}, \\ \dfrac{\sqrt{2}}{\sqrt{p+q+2}} \cot \dfrac{\pi j}{2(p+q+2)}, & j \text{ odd}, \end{cases}$$

and therefore[5]

Prob. $\{M > n\}$

$$= \frac{2}{p+q+2} \sum_{j=1}^{p+q+1}{}^{*} \cos^n \frac{\pi j}{p+q+2} \sin \frac{\pi j(q+1)}{p+q+2} \cot \frac{\pi j}{2(p+q+2)},$$

where the star on the summation sign indicates that only odd j's are taken under account.

The method just illustrated is quite general but in more complicated cases the job of finding the eigenvalues and eigenvectors becomes formidable.

[5] Professor Feller has called the author's attention to the fact that similar problems and formulas can be found in Chapter III of W. Burnside's *Theory of Probability* (Cambridge, 1928). He also pointed out that the problem could be treated by means of Markoff chains.

Professor G. E. Uhlenbeck has pointed out that our formula implies a known result from the theory of Brownian motion.

Consider a free Brownian particle which at $t = 0$ is at $x = x_0 (x_0 > 0)$. R. Fürth[6] has shown that the probability that between t and $t + dt$ the particle will be either at $x = 0$ or at $x = d$ $(0 < x_0 < d)$ for the first time, is given by the formula

$$dt \frac{4\pi D}{d^2} \sum_{m=0}^{\infty} (2m + 1)e^{(-\pi^2 Dt/d^2)(2m+1)^2} \sin \frac{(2m + 1)\pi x_0}{d},$$

where D is the "coefficient of diffusion"

If we treat the one-dimensional Brownian motion as a random walk with steps $\pm\Delta x$, each move lasting Δt, the probability that a particle starting from x_0 will not have reached 0 or d in the time interval $(0, t)$ can be calculated by means of our formula.

We must only put $q = x_0/\Delta x$, $p = (d - x_0)/\Delta x$, $n = t/\Delta t$ and assume that as both Δx and Δt approach 0 the ratio $(\Delta x)^2/2\Delta t$ approaches the "coefficient of diffusion" D.

An elementary computation shows that in this limit the Prob. $\{N > t/\Delta t\}$ approaches

$$\frac{4}{\pi} \sum_{j=1}^{\infty}{}^{*} \frac{1}{j} e^{(-\pi^2 j^2 D/d^2)t} \sin \frac{\pi j x_0}{d}$$

and that the differential of this expression (with a minus sign) gives exactly Fürth's expression.

3. General theory in the continuous case. We now assume that the distribution function of X possesses a continuous and even density function $\rho(x)$. We have

$$\text{Prob. } \{N > n\} = \int \cdots \int_{\Omega} \rho(x_1) \cdots \rho(x_n)\, dx_1 \cdots dx_n,$$

where the region of integration Ω is defined by the inequalities

$$-q \leq x_1 \leq p, \qquad -q \leq x_1 + x_2 \leq p, \cdots, \qquad -q \leq x_1 + \cdots + x_n \leq p$$

Introducing the new variables

$$y_j = q + x_1 + \cdots + x_j, \qquad (j = 1, 2, \cdots, n),$$

we see that the Jacobian of the transformation is 1 and

$$\text{(3)} \quad \begin{aligned} &\text{Prob. } \{N > n\} \\ &= \int_0^{p+q} \cdots \int_0^{p+q} \rho(y_1 - q)\rho(y_2 - y_1) \cdots \rho(y_n - y_{n-1})\, dy \cdots dy_n . \end{aligned}$$

Consider the symmetric integral equation

$$\text{(4)} \qquad \int_0^{p+q} \rho(s - t) f(t)\, dt = \lambda f(s)$$

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[6] *Ann. d. Phys.* 53 (1917) p 177.

and note that if $K_n(s, t)$ denotes the n-th iterated kernel of this integral equation, the right side of (3) is equal to

$$\int_0^{p+q} K_n(q, t)\, dt.$$

Thus

$$\text{Prob. } \{N > n\} = \int_0^{p+q} K_n(q, t)\, dt.$$

From the general theory of integral equations we know that

$$K_n(s, t) = \sum_{j=1}^{\infty} \lambda_j^n f_j(s) f_j(t), \qquad (n \geq 2),$$

where $\lambda_1, \lambda_2, \cdots$ are eigenvalues and $f_1(t), f_2(t), \cdots$ normalized eigenfunctions of the integral equation (4).

Since ρ was assumed to be continuous it follows that the eigenfunctions are continuous and

$$\text{Prob. } \{N > n\} = \sum_{j=1}^{\infty} \lambda_j^n f_j(q) \int_0^{p+q} f_j(t)\, dt.$$

This formula is very general and provides, in a sense, a complete solution of the problem in the continuous and symmetric case. Unfortunately the usefulness of this formula is limited by the difficulties encountered in solving integral equations of the type (4).

In fact, the integral equation

$$\frac{1}{\sqrt{2\pi}} \int_0^a e^{-(s-t)^2/2} f(t)\, dt = \lambda f(s),$$

to which one is led by considering the normally distributed X's, appears to be very difficult to solve.

4. A particular case. If we assume

$$\rho(x) = \frac{1}{2} e^{-|x|}$$

we are led to the integral equation

$$\int_0^{p+q} e^{-|s-t|} f(t)\, dt = 2\lambda f(s),^{7} \tag{5}$$

which is quite easy to solve.

In fact, rewriting (5) in the form

$$e^{-s} \int_0^{p+q} e^{t} f(t)\, dt + e^{s} \int_0^{p+q} e^{-t} f(t)\, dt = 2\lambda f(s) \tag{6}$$

[7] I have recently encountered the integral equation (5) in solving an entirely different problem. A complete discussion can be found in a restricted N D.R.C. Report 14-305.

and differentiating twice with respect to s we obtain the differential equation

$$f''(s) + \left(\frac{1}{\lambda} - 1\right) f(s) = 0.$$

Substituting the general solution of this equation in (6) we find in an entirely elementary fashion that

$$\lambda_j = \frac{1}{1 + y_j^2},$$

$$f_j(t) = \frac{\sin y_j t + y_j \cos y_j t}{\sqrt{1 + \frac{1}{2}(p + q)(1 + y_j^2)}},$$

where y_j is the jth (positive) root of the transcendental equation

$$\tan (p + q)y = -\frac{2y}{1 - y^2}. \tag{7}$$

We have

$$\int_0^{p+q} (\sin y_j t + y_j \cos y_j t)\, dt = \frac{1}{y_j} \{1 - \cos (p + q)y_j + y_j \sin (p + q)y_j\}$$

and it is easily seen that (7) implies

$$1 - \cos (p + q)y_j + y_j \sin (p + q)y_j = \begin{cases} 0 & \text{if } \cos (p + q)y_j = \dfrac{1 - y_j^2}{1 + y_j^2}, \\ 2 & \text{if } \cos (p + q)y_j = -\dfrac{1 - y_j^2}{1 + y_j^2}. \end{cases}$$

Finally,

$$\text{Prob. } \{N > n\} = 2 \sum_{j=1}^{\infty}{}' \frac{1}{(1 + y_j^2)^n} \frac{\sin y_j q + y_j \cos y_j q}{y_j \{1 + \frac{1}{2}(p + q)(1 + y_j^2)\}},$$

where the dash on the summation sign indicates that only those j's are taken under account for which

$$\cos (p + q)y_j = -\frac{1 - y_j^2}{1 + y_j^2}.$$

We omit here the discussion of various limiting cases inasmuch as our main purpose was to obtain exact formulas.

There are indications that some of the limiting cases are related to singular integral equations with continuous spectra. We may return to this subject at a later date.

ON THE CLASSIFICATION OF OBSERVATION DATA INTO DISTINCT GROUPS

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Introduction. In scholastic examinations as well as in the examination of industrial products the following probability problem arises. The individuals of a certain population are successively subjected to trials each of which leads to a definite score x (one real number or a group of m real numbers). Each individual is supposed to belong to one of n classes. These classes are characterized by n probability densities $p_1(x)$, $p_2(x)$, $\cdots$ $p_n(x)$. One has to decide on the basis of the observed value x to which class the respective individual belongs and one wishes to make this decision with the smallest possible risk of failure.

For example, let us consider an examination where the three grades A, B, C are attributed on the basis of a simple score x (case $m = 1, n = 3$). It may be assumed that an individual of the class A has a mean expected value of x equal to $\vartheta_1 = 75$ and a normal distribution with the standard deviation $\sigma_1 = 4/\sqrt{2}$. The analogous values for the classes B and C may be $\vartheta_2 = 50$, $\sigma_2 = 8/\sqrt{2}$ and $\vartheta_3 = 25$, $\sigma_3 = 12/\sqrt{2}$. In this case, the solution developed in the present paper allows the conclusion that the best way of grading would be to attribute the grade A to scores x beyond 70.0, the grade C to scores below 40.0 and B to the rest. The corresponding error risk will be 3.9% or the success rate 0.961.

There exists, of course, one case where the solution is trivial. If the probability densities $p_\nu(x)$ are limited to n non-overlapping regions R_ν (with $p_\nu = 0$ at points outside R_ν) an obvious decision can be made without any risk of failure. An assumption of this kind underlies the usual procedure of grading. If, in the foregoing example, an individual of class A is supposed to have at any rate a score beyond 60 and a class C individual less than 40, it is obvious how the grades should be attributed without incurring any risk. It seems, however, that in many problems the assumption of normal distributions or some other kind of overlapping distributions is more appropriate. Then, the probability problem has to be solved.

The solution submitted in the present paper is derived from the simplest principles of calculus of probability without any arbitrary assumption or hypothesis. If n equals 2, the problem can also be considered as a problem of testing a simple statistical hypothesis with a two-valued parameter.[1] It has been shown in an earlier paper[2] that under this restriction success rates higher than 50% are obtainable.

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[1] See A. Wald, *Annals of Math. Stat.*, Vol. 15 (1944), p. 145. Here, both $p_1(x)$ and $p_2(x)$ are supposed to be normal distributions with the same covariance matrix. The problem treated by Wald is different from the one considered in the present paper since in Wald's paper the parameters of the two multivariate normal distributions are assumed to be unknown.

[2] R. v. Mises, *Annals of Math Stat.*, Vol 14 (1943), p. 238.

1. Statement of the problem. For each of n classes of individuals a probability density $p_\nu(x)$, $\nu = 1, 2, \cdots n$, is given. We subdivide the m-dimensional x-space into n regions $R_1, R_2, \cdots R_n$ and assign the region R_ν to the νth class. The probability, for an individual of this class, to have its x-value falling in R_ν is

$$P_\nu = \int_{(R_\nu)} p_\nu(x)\, dX, \qquad \nu = 1, 2, \cdots n \tag{1}$$

where dX denotes the element of the x-space ($dX = dx$ in the case $m = 1$). In the N first trials of the indefinite sequence of trials, N_ν individuals that belong to the νth class will be tested. Out of these only those individuals whose x-value falls in R_ν will be ascribed to the νth class. Their number according to the definition of probability, equals $N_\nu(P_\nu + \epsilon_\nu)$ where ϵ_ν tends towards zero as N_ν goes to infinity. The total number of correct decisions during the N first trials is therefore

$$N_1(P_1 + \epsilon_1) + N_2(P_2 + \epsilon_2) \cdots N_n(P_n + \epsilon_n) \tag{2}$$

and the relative number is

$$\frac{N_1}{N}(P_1 + \epsilon_1) + \frac{N_2}{N}(P_2 + \epsilon_2) + \cdots \frac{N_n}{N}(P_n + \epsilon_n). \tag{2'}$$

If N increases indefinitely a part of the N_ν must become infinite. For these classes, ϵ_ν converges toward zero. For the other classes N_ν/N diminishes to zero. Thus, the relative number of right decisions converges towards

$$\frac{1}{N}(N_1P_1 + N_2P_2 + \cdots N_nP_n). \tag{3}$$

The N_ν are unknown. Every one of these unknowns can take each value from zero to N. If P_μ is the smallest P_ν, the most unfavorable case, where the expression (3) has its smallest value, will occur with $N_\mu = N$, all other N_ν being zero. This value is obviously P_μ. Thus it is seen that the frequency of correct assignments is at least equal to the smallest P_ν which may be written as $P_{\min}$. The greatest risk of making a false decision is $1 - P_{\min}$.

Now the problem to be solved in the present paper can be stated as follows: *For n given densities $p_\nu(x)$, find the subdivision of the x-space into n regions R_ν that gives to the smallest of the expressions P_ν defined in* (1) *its possibly greatest value.*

This problem has the type of a continuous variation problem with the integrals in question bounded within the limits zero to one. We may, therefore, assume that under reasonable restrictions for $p_\nu(x)$ a solution exists. Uniqueness of the solution cannot be expected in general. It seems very difficult to establish the conditions for unicity in other than the most simple cases. Existence of more than one solution would mean that each of them is an optimum with respect to infinitesimal modifications of the boundaries.

2. General solution. A simple problem of variation is considered as solved in principle when the nature of the extremals is known. In our case of a so-called minimax problem, where the minimum of n quantities is maximized, an additional relation between the n integrals is required. Both can easily be found in the actual case.

Let us first consider a partition of the x-space into n regions with not all P_ν being equal. The smallest P_ν will be called $P_{\min}$ and the smallest but one P^*. Among the k regions for which $P_\nu = P_{\min}$ there will be at least one, say, R_α that has a common border with a region R_β whose P-value is greater, so that $P_\beta \geqq P^*$. Now modify the boundary between R_α and R_β in such a way that the space covered by R_α is increased and that of R_β decreased. According to (1) the new values of P_α and P_β will be

$$P'_\alpha = P_\alpha + \Delta, \qquad P'_\beta = P_\beta - \Delta' \tag{4}$$

with both Δ and Δ' positive The two quantities Δ and Δ' are not independent of one another, but they can be chosen both smaller than any given positive number ϵ. Therefore, the condition

$$P'_\alpha = P_\alpha + \Delta < P_\beta - \Delta' = P'_\beta \tag{5}$$

can be fulfilled. All other P_ν-values remain unchanged.

In the case $k = 1$, that is, if only one region R_ν had originally the minimum P-value, the modified system has a greater minimum P, which equals either $P_\alpha + \Delta$ or P^*. If $k > 1$ the new system has the same minimum P as the original one, but its k-value is diminished by one. If we repeat the same procedure $(k - 1)$ times we obtain a system of regions with one single P_ν having the minimum P-value and the next step leads to a partition of the x-space into n regions with a smallest P-value that is greater than the original $P_{\min}$. Thus it is seen that no partition with unequal P_ν-values can solve our problem.

Secondly, if $m > 1$, consider a system of n regions with $P = P_1 = P_2 = \cdots = P_n$. Take two points, x and y, on the border of any two neighboring regions R_ν and R_μ. An infinitesimal variation of the boundary would consist of adding to R_ν in the neighborhood of the point x a space element δS subtracting it from R_μ and, at the same time, adding to R_μ in the vicinity of y an element $\delta S'$ subtracting it from R_ν. Then, according to (1), the new values of P_ν and P_μ will be

$$\begin{aligned} P'_\nu &= P + p_\nu(x)\delta S - p_\nu(y)\delta S' \\ P'_\mu &= P - p_\mu(x)\delta S + p_\mu(y)\delta S'. \end{aligned} \tag{6}$$

Introducing $\Delta_\nu = P'_\nu - P$ and $\Delta_\mu = P'_\mu - P$, these equations solved for δS and $\delta S'$ give

$$\delta S = \frac{p_\nu(y)\Delta_\mu + p_\mu(y)\Delta_\nu}{D}, \qquad \delta S' = \frac{p_\nu(x)\Delta_\mu + p_\mu(x)\Delta_\nu}{D} \tag{7}$$

where

$$D = p_\nu(x)p_\mu(y) - p_\mu(x)p_\nu(y). \tag{7'}$$

If the determinant D is positive, we find two positive quantities δS and $\delta S'$ for any pair of positive Δ_μ and Δ_ν. If D is negative the same is true when x and y are interchanged. In both cases, that is, with $D \neq 0$, the original partition is replaced by a new system of regions in which only two regions, R_ν and R_μ, have increased P-values, while (if $n > 2$) still $P_{min} = P$. If to this system the procedure as described in the foregoing is applied, a final partition with a greater minimum value of P can be derived. The conclusion is that no solution of our problem can include a boundary on which the determinant D is different from zero for any two points x and y. On the other hand, it is seen that $D = 0$ means that the ratio $p_\nu(x):p_\mu(x)$ has a constant value along the border. Thus the result is reached:

The partition of the x-space that solves our problem is characterized by two properties: (1) *for all n regions R_ν the value of P_ν is the same;* (2) *along the border between R_ν and R_μ the ratio $p_\nu(x)/p_\mu(x)$ is constant.*

In the one-dimensional case ($m = 1$) only the first of these two statements is relevant. In any case, the success rate, that is, the guaranteed ratio of correct decisions, equals the common value of all P_ν.

3. Illustrations. (a) One-dimensional case. Upon introducing the cumulative distribution functions

$$F_\nu(x) = \int_{-\infty}^{x} p_\nu(z)dz \tag{8}$$

the conditions $P_1 = P_2 = \cdots P_n$ take the form

$$F_1(x_1) = F_2(x_2) - F_2(x_1) = \cdots = F_{n-1}(x_{n-1}) - F_{n-1}(x_{n-2}) = 1 - F_n(x_{n-1}) \tag{9}$$

where $x_1, x_2, \cdots x_{n-1}$ determine the n intervals on the both-sides infinite x-axis. If all density functions have the same form except for an affine transformation one has

$$F_\nu(x) = F[h_\nu(x - \vartheta_\nu)], \qquad \nu = 1, 2, \cdots n \tag{10}$$

Let us assume, for instance, that scores between 0 and 100 are attributed to three types of individuals. The first type may have an even chance to obtain a score between 0 and 50, the second between 40 and 80 and the third between 70 and 100. Here

$$F_\nu(x) = \tfrac{1}{2} + (x - \vartheta_\nu)p_\nu, \qquad |x - \vartheta_\nu| \leq \frac{1}{2p_\nu} \tag{11}$$

with $\vartheta_\nu = 25, 60, 85$ and $p_\nu = \frac{1}{50}, \frac{1}{40}, \frac{1}{30}$. The conditions (9) supply

$$\frac{1}{2} + \frac{x_1 - 25}{50} = \frac{1}{40}(x_2 - x_1) = \frac{1}{2} - \frac{x_2 - 85}{30} \tag{12}$$

and this, solved for x_1, x_2 gives $x_1 = 41\frac{2}{3}$, $x_2 = 75$ while the three expressions (12) take the value 0.833. Therefore, in attributing all scores below $41\frac{2}{3}$ to the first class and all scores beyond 75 to the third one is safe to make under no circumstances more than $\frac{1}{6}$ incorrect decisions in the long run.

In the example quoted in the introduction one has

$$p_\nu(x) = \frac{1}{\sigma_\nu\sqrt{2\pi}}\, e^{(x-\vartheta_\nu)^2/2\sigma_\nu^2} \tag{13}$$

with $\vartheta_\nu = 75, 50, 25$ and $\sigma_\nu^2 = 8, 32, 72$. If $\Phi(x)$ denotes the integral

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz$$

the conditions (9) become

$$1 + \Phi\left(\frac{x_1 - 25}{12}\right) = \Phi\left(\frac{x_2 - 50}{8}\right) - \Phi\left(\frac{x_1 - 50}{8}\right) = 1 - \Phi\left(\frac{x_2 - 75}{4}\right). \tag{14}$$

The first and last expression equated lead to $x_1 + 3x_2 = 250$. The complete solution can be found with the help of tables for Φ. It is $x_1 = 29.9920$, $x_2 = 70\,0027$ with the common value twice 0.961 for the three expressions (14). Hence the result as quoted in the introduction.

Let us now take up the case of six normal distributions with equidistant mean values $\vartheta = \pm a, \pm 3a, \pm 5a$ and one and the same variance σ^2. Then, because of symmetry, two equations only have to be fulfilled:

$$1 + \Phi\left(\frac{x_1 + 5a}{\sigma\sqrt{2}}\right) = \Phi\left(\frac{x_2 + 3a}{\sigma\sqrt{2}}\right) - \Phi\left(\frac{x_1 + 3a}{\sigma\sqrt{2}}\right) = \Phi\left(\frac{a}{\sigma\sqrt{2}}\right) - \Phi\left(\frac{x_2 + a}{\sigma\sqrt{2}}\right)$$

For $\sigma^2/a^2 = 0.32$, the numerical solution gives

$$x_1 = -4.160a, \qquad x_2 = -2.062a.$$

The success rate, i.e. half the common value of the above expressions is 0.931. The six intervals extend from $-\infty$ to x_1, from x_1 to x_2, from x_2 to 0, from 0 to $-x_2$, from $-x_2$ to $-x_1$, and from $-x_1$ to ∞.

(b) *Case of more than one dimension.* Let us assume that two classes A and B have uniform distributions extending over volumes $V_1 = 1/p_1$ and $V_2 = 1/p_2$ respectively. If the two regions have a common part of volume V each surface within the common space fulfills the condition $p_1/p_2 =$ constant. Thus, the two regions R_1 and R_2 are not uniquely determined but subject to one condition only which determines the optimum success rate. If κV is cut out from V_1 and $(1 - \kappa)V$ from V_2, the relation must be fulfilled:

$$1 - p_1 V\kappa = 1 - p_2 V(1 - \kappa), \qquad i.e.\ \kappa = \frac{p_2 V}{p_1 + p_2}$$

and the success rate is

$$S = 1 - p_1 V\kappa = 1 - \frac{p_1 p_2 V}{p_1 + p_2} = 1 - p_2 V(1 - \kappa).$$

If three classes A, B, and C are considered with the densities $p_1 = 1/V_1$, $p_2 = 1/V_2$, $p_3 = 1/V_3$ and the first two regions have a space of volume V in common, the latter two a space of volume V', the conditions are

$$1 - p_1 V(1 - \kappa) = 1 - p_2(\kappa V + \lambda V') = 1 - p_3(1 - \lambda)V'$$

which supply

$$\kappa = 1 - \frac{p_2 + p_3}{p_1 p_2 + p_2 p_3 + p_3 p_1} \frac{V + V'}{V},$$

$$\lambda = 1 - \frac{p_1 p_2 p_3}{p_1 p_2 + p_2 p_3 + p_3 p_1} \frac{V + V'}{V'}.$$

and the success rate has the value

$$S = 1 - (V + V') \frac{p_1 p_2 p_3}{p_1 p_2 + p_2 p_3 + p_3 p_1}.$$

If the p_ν are normal density functions, say

$$p_\nu(x, y) = \frac{\sqrt{D_\nu}}{\pi} e^{-\frac{1}{2} Q_\nu},$$

$$Q_\nu = \alpha_\nu (x - a_\nu)^2 + 2\beta_\nu (x - a_\nu)(y - b_\nu) + \gamma_\nu (y - b_\nu)^2$$

and D_ν the corresponding determinants, the curves separating the regions R_ν are the conics

$$Q_\nu - Q_\mu = \text{const}.$$

where the constants are determined by the conditions that all P_ν must be equal. If the α, β, γ have the same values for every ν, the borders consist of straight lines. In this case one can reduce the expressions for p_ν, by an affine transformation, to

$$p_\nu(x, y) = \frac{1}{\pi} e^{-(x - a_\nu)^2 - (y - b_\nu)^2}.$$

In the transformed plane the borderline between the regions R_ν and R_μ is perpendicular to the straight line that connects the points $A_\nu(a_\nu, b_\nu)$ and $A_\mu(a_\mu, b_\mu)$. If all points A_ν lie on the same straight line (in particular, if $n = 2$) the whole problem is practically identical with the one-dimensional ($m = 1$). In the case $n = 3$, in general, the three regions are confined by three lines perpendicular to A_1A_2, A_2A_3, A_3A_1 passing through a point C whose coordinates are determined by the equations $P_1 = P_2 = P_3$. If r_ν denotes the distance $A_\nu C$ and φ_ν, ϑ_ν are the angles, $A_\nu C$ forms with the adjacent sides of the triangle $A_1A_2A_3$ one has to use the function

$$F(r, \varphi_1) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \phi(r - z \tan \varphi) e^{-z^2} \, dz.$$

Then the two conditions for C read

$$F(r_1, \varphi_1) + F(r_1, \vartheta_1) = F(r_2, \varphi_2) + F(r_2, \vartheta_2) = F(r_3, \varphi_3) + F(r_3, \vartheta_3)$$

and the success rate equals 0.5 plus the common value of these three expressions.

ON AN EXTENSION OF THE CONCEPT OF MOMENT WITH APPLICATIONS TO MEASURES OF VARIABILITY, GENERAL SIMILARITY, AND OVERLAPPING[1]

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1. Introduction. Given a frequency distribution D: $[X_i, F_i]$ ($i = 1, 2, 3, \cdots, n$), we shall call the expression

$$M_r(D, X_j) = \sum_{i=1}^{n} (X_i - X_j)^r F_i$$

the rth total moment of D about the origin X_j. We shall consider the weighted sum

$$\mathfrak{M}_r = \Sigma_j W_j M_r(D, X_j)$$

where W_j denotes the weight corresponding to the particular origin X_j, and the summation is over a field ϕ. In particular, if ϕ is the set of all values assumed in D by the variate X_i, and if $W_j = F_j$, we shall call the quantity the rth *complete total moment* of D. If, on the contrary, W_j is the frequency F'_j of the value X'_j in a second frequency distribution D': $[X'_j, F'_j]$ and ϕ' is the set of all values assumed by the variate X'_j in D', $\mathfrak{M}_r$ will be called the rth *aggregate moment* of D and D'. A modification of this procedure leads to what we shall call the *moment of transvariation* of D and D'.

The consideration of complete moments draws attention to certain previously known measures of variability which are independent of the origin selected, and also provides simple methods of computation which are useful for data given in the form of a frequency distribution. The investigation of aggregate moments and moments of transvariation gives rise to certain measures of general similarity between two distributions, as well as measures of the amount of overlapping.

2. Sliding and complete moments of a frequency distribution.

2.1. We shall give the name *sliding total moments* of order r to the successive values, for particular values of j, of the expression

$$M_r(X_j) = F_j \sum_{i=1}^{n} [(X_i - X_j)^r F_i]. \tag{2.11}$$

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[1] The Portuguese original of this paper was written in Brazil, in August 1943. Its translation into English was entirely revised by Dr. T. Greville, Bureau of the Census, who proposed also many simplifications in the deriyation of formulae. For his painstaking labor and interest I wish to express my very sincere appreciation. I also wish to thank Dr. W. Edwards Deming for reading the manuscript and making several valuable suggestions.

The expression for the complete total moment, written out in full, is

$$\mathfrak{M}_r = \sum_{j=1}^{n} M_r(X_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} [(X_i - X_j)^r F_i F_j]. \tag{2.12}$$

It is readily seen that the complete moment is independent of the choice of origin.

2.2. If $r = 0$, we have

$$M_0(X_j) = F_j \sum_{i=1}^{n} F_i .$$

The complete total moment of order zero will therefore be

$$\mathfrak{M}_0 = \sum_{j=1}^{n} F_j \sum_{i=1}^{n} F_i = M_0^2 \tag{2.21}$$

where M_0 stands for the total moment of order zero about the origin of the X', that is,

$$M_0 = N\nu_0' .$$

2.3. If $r = 1$, we shall have

$$M_1(X_j) = F_j \sum_{i=1}^{n} [(X_i - X_j) F_i].$$

Using M_1 to denote the total moment of order one about the origin of the X, we obtain

$$M_1(X_j) = F_j \sum_{1}^{n} X_i F_i - X_j F_j \sum_{1}^{n} F_i = F_j M_1 - X_j F_j M_0 .$$

Making j vary from 1 to n and summing, we have

$$\begin{aligned} \mathfrak{M}_1 &= \sum_{j=1}^{n} F_j M_1 - \sum_{j=1}^{n} X_j F_j M_0 \\ &= M_0 M_1 - M_1 M_0 = 0 \end{aligned} \tag{2.31}$$

This result is due to the fact that we took the deviations $X_i - X_j$ with their proper signs. We may, however, calculate the value which the complete moment of first order would have if using absolute values. Thus, the sliding total moment thus modified becomes

$$|M_1(X_j)| = F_j \left[\sum_{i=1}^{j-1} (X_j - X_i) F_i + \sum_{i=j}^{n} (X_i - X_j) F_i \right]$$

which may be put in the form

$$|M_1(X_j)| = F_j X_j \left[\sum_{i=1}^{j-1} F_i - \sum_{i=j}^{n} F_i \right] - F_j \left[\sum_{i=1}^{j-1} F_i X_i - \sum_{i=j}^{n} F_i X_i \right]. \tag{2.32}$$

Summing with respect to j and employing the substitutions

$$(2.33)\qquad \begin{aligned} \sum_{i=j}^{n} F_i &= M_0 - \sum_{i=1}^{j-1} F_i \\ \sum_{i=j}^{n} F_i X_i &= M_1 - \sum_{i=1}^{j-1} F_i X_i \end{aligned}$$

gives for the complete total moment

$$(2.34)\qquad |\mathfrak{M}_1| = 2\sum_{j=1}^{n}\left[F_j X_j \sum_{i=1}^{j-1} F_i\right] - 2\sum_{j=1}^{n}\left[F_j \sum_{i=1}^{j-1} F_i X_i\right].$$

The quotient

$$(2.35)\qquad m_1 = \frac{|\mathfrak{M}_1|}{\mathfrak{M}_0}$$

of the complete total moment of order one by the complete total moment of order zero we shall call the *complete unit moment* of order one, or simply the complete moment of order one, when no confusion would result.

The complete unit moment is a measure of variability, identical with that already considered by Andrae and Helmert, respectively in 1869 and in 1876, and which C. Gini, in 1912, called mean difference with repetition.[2]

The numerator of m_1 is easily computed if we observe that the upper limit $j - 1$ of the F_i summation, for example, means that each product $X_j F_j$ must be multiplied by the cumulative frequency corresponding to the class immediately preceding. We only have to shift the cumulative frequencies column by one class in the proper direction; the second term is similarly dealt with.

2.4. The second order sliding total moment is

$$M_2(X_j) = F_j \sum_{i=1}^{n} [(X_i - X_j)^2 F_i] = F_j M_2 - 2F_j X_j M_1 + F_j X_j^2 M_0$$

where M_2 is the total moment of order two. Summing with respect to j gives the complete total moment of order two

$$(2.41)\qquad \mathfrak{M}_2 = \sum_{j=1}^{n} M_2(X_j) = 2(M_2 M_0 - M_1^2).$$

The complete unit moment of order two is therefore

$$(2.42)\qquad \begin{aligned} m_2 &= 2\left[\frac{M_2}{M_0} - \left(\frac{M_1}{M_0}\right)^2\right] = 2(\nu_2' - \nu_1'^2) \\ &= 2\sigma^2 \end{aligned}$$

[2] Apud Czuber, *Wahrscheinlichkeitsrechnung*, Vol. 2, (1932), p. 316. C. Gini, *Variabilità e Mutabilità*, Cagliari, 1912

where ν' stands for a unit moment about the origin of the X, namely

$$\nu'_r = \frac{\Sigma X^r F}{\Sigma F},$$

m_2 is also a measure of variability, independent of the choice of origin. It is equal to the square of Gauss's "Präzisionsmass", and to the double of Fisher's variance; like m_1 it was defined by Andrae and Helmert, and was called by Gini the mean square difference with repetition.

2.5. If $r = 3$ we have for the sliding moments,

$$\begin{aligned} M_3(X_j) &= F_j \sum_{i=1}^{n} (X_i - X_j)^3 F_i \\ &= F_j M_3 - 3F_j X_j M_2 + 3F_j X_j^2 M_1 - F_j X_j^3 M_0 . \end{aligned}$$

Summation over j gives

$$\text{(2.51)} \quad \mathfrak{M}_3 = \sum_{j=1}^{n} M_3(X_j) = M_0 M_3 - 3M_1 M_2 + 3M_2 M_1 - M_3 M_0 = 0,$$

a result which is easily shown to hold for any complete moment of odd order. We may calculate the value of the complete moment of order three using absolute values of the deviations $X_i - X_j$ by a process similar to that previously described for the calculation of $|\mathfrak{M}_1|$. This gives

$$\text{(2.52)} \quad \begin{aligned} |\mathfrak{M}_3| = 2\Bigg[&\sum_{j=1}^{n} F_j X_j^3 \sum_{i=1}^{j-1} F_i - 3 \sum_{j=1}^{n} F_j X_j^2 \sum_{i=1}^{j-1} F_i X_i \\ &+ 3 \sum_{j=1}^{n} F_j X_j \sum_{i=1}^{j-1} F_i X_i^2 - \sum_{j=1}^{n} F_j \sum_{i=1}^{j-1} F_i X_i^3 \Bigg]. \end{aligned}$$

2.6. The sliding moments of order four are

$$M_4(X_j) = F_j M_4 - 4F_j X_j M_3 + 6F_j X_j^2 M_2 - 4F_j X_j^3 M_1 + F_j X_j^4 M_0 .$$

Summing with respect to j and simplifying, we have

$$\text{(2.61)} \quad \begin{aligned} \mathfrak{M}_4 &= M_0 M_4 - 4M_1 M_3 + 6M_2^2 - 4M_3 M_1 + M_4 M_0 \\ &= 2(M_0 M_4 - 4M_1 M_3 + 3M_2^2). \end{aligned}$$

Dividing both sides by $\mathfrak{M}_0$ in order to obtain the complete moment on a unit basis, we have

$$m_4 = 2\left[\frac{M_4}{M_0} - 4\frac{M_1}{M_0}\frac{M_3}{M_0} + 3\left(\frac{M_2}{M_0}\right)^2\right] = 2\,(\nu'_4 - 4\nu'_1\nu'_3 + 3\nu'^2_2).$$

But, if ν indicates a moment about the mean

$$\nu_4 = \nu'_4 - 4\nu'_1\nu'_3 + 6\nu'^2_1\nu'_2 - 3\nu'^4_1.$$

By substitution, therefore

$$
\begin{aligned}
m_4 &= 2(\nu_4 + 3\nu_2'^2 - 6\nu_1'^2\nu_2' + 3\nu_1'^4) \\
&= 2[\nu_4 + 3(\nu_2' - \nu_1'^2)^2] \qquad (2.62)\\
&= 2(\nu_4 + 3\nu_2^2).
\end{aligned}
$$

This complete moment gives rise to a measure of kurtosis independent of the choice of origin

$$k = \frac{m_4}{m_2^2} = \frac{\nu_4}{2\nu_2^2} + \frac{3}{2}.$$

In case of mesokurtosis this reduces to 3, since for the normal curve $\nu^4 / \nu_2^2 = 3$; leptokurtosis and platikurtosis occur for the same ranges as in the case of Pearson's measure β_2.

3. Aggregate moments of two frequency distributions.

3.1. Given two frequency distributions, D:$[X_i, F_i](i = 1, 2, 3, \cdots, n)$ and D': $[X_j', F_j'](j = 1, 2, 3, \cdots, p)$ and a fixed point X_j' belonging to the second distribution, we shall call the expression

$$M_r(D, X_j') = F_j' \sum_{i=1}^{n} (X_i - X_j')^r F_i \tag{3.11}$$

the rth *aggregate sliding total moment* of the first distribution about the element X_j' of the second. Summation over j gives

$${}^c\mathfrak{M}_r = \sum_{j=1}^{p} \sum_{i=1}^{n} F_j'(X_i - X_j')^r F_i. \tag{3.12}$$

We shall call ${}^c\mathfrak{M}_r$ the *aggregate complete total moment* or, simply, the aggregate total moment of D about D'. It is clear that this is a symmetric function of the two distributions, except for a change of sign in the case of odd moments.

3.2. If $r = 0$, we have

$$M_0(D, X_j') = F_j' \sum_{i=1}^{n} F_i \tag{3.21}$$

$${}^c\mathfrak{M}_0 = \sum_{j=1}^{p} F_j' \sum_{i=1}^{n} F_i = M_0 M_0'. \tag{3.22}$$

3.3. If $r = 1$, we have

$$M_1(D, X_j') = F_j' M_1 - F_j' X_j' M_0 \tag{3.31}$$

$${}^c\mathfrak{M}_1 = M_1 M_p' - M_0 M_1'. \tag{3.32}$$

We shall call the quotient

$${}^c m_1 = \frac{{}^c\mathfrak{M}_1}{{}^c\mathfrak{M}_0} \tag{3.33}$$

the aggregate *unit* moment of order r (or the aggregate moment coefficient), or simply the aggregate moment of order r whenever the simpler name will not cause confusion.

It is obvious that the aggregate moments are measures of general similarity, as to form and position, between D and D'. This similarity will be an identity in case the two distributions coincide perfectly; on the other hand, it is clear that there is no limit to the degree of non-similarity which may be encountered. We shall take unity to represent the maximum and zero the minimum of similarity, and thus define a provisional similarity index

$$S = \frac{m_1 m_1'}{{}^c m_1^2}. \tag{3.34}$$

But

$${}^c m_1 = \frac{M_1 M_0' - M_0 M_1'}{M_0 M_0'} = A - A'$$

where A and A' stand for the arithmetic means of D and D', respectively. Now it will be seen that if $A = A'$, $S = \infty$. This result is due to the fact that in the calculation of m_1 and m_1' we took the absolute values of the deviations $X_i - X_j'$, while in the calculation of ${}^c m_1$ we retained the algebraic signs. In order to make the two terms of the fraction in (3.34) comparable, we can either: 1) calculate ${}^c m_1$ also using absolute values; or 2) take only the positive or only the negative part of both numerator and denominator of S. In any case, $A = A'$ is a necessary condition for the maximum of S.

3.4. We shall employ the first method suggested above, although we shall return to the second in the third part of the paper. As long as D and D' do not overlap, all the $X_i - X_j'$ deviations have the same sign and this is the same as that of the difference $A - A'$. If, however, there is some overlapping this will not be the case, some deviations having different signs from that of $A - A'$. This brings us to Gini's concept of "transvariation". He applies this term to any deviation $X_i - X_j'$ which does not have the same sign as $\bar{X} - \bar{X}'$, these symbols denoting averages of any previously specified type; and he calls the magnitude of the deviation its "intensity".

In computing the complete moment of the first order using absolute values, in order to simplify the algebra we shall assume the same origin for X and X' and therefore drop the stroke from the X, but not of course from the F. If certain values of X occur in one distribution and not in the other, we can merely consider the frequency as zero in the second distribution. In this way the two distributions can be regarded as extending over the same total range. If X_1 and X_m denote the extreme values, the sliding total moment is

$$\begin{aligned} |M_1(D, X_j)| &= F_j' \left[\sum_{i=1}^{j-1} (X_j - X_i)F_i + \sum_{i=j}^{m} (X_i - X_j)F_i \right] \\ &= F_j' X_j \left(\sum_{i=1}^{j-1} F_i - \sum_{i=j}^{m} F_1 \right) - F_j' \left(\sum_{i=1}^{j-1} F_i X_i - \sum_{i=j}^{m} F_i X_i \right). \end{aligned}$$

Summing with respect to j and at the same time employing the substitutions (2.33) or their transposed form, we obtain the following alternative expressions for the complete aggregate moment:

$$(3.41)\quad |{}^c\mathfrak{M}_1| = M_1M_0' - M_0M_1' + 2\sum_{j=1}^{m}\left[F_j'X_j\sum_{i=1}^{j-1}F_i\right] - 2\sum_{j=1}^{m}\left[F_j'\sum_{i=1}^{j-1}F_iX_i\right]$$

$$(3.42)\quad |{}^c\mathfrak{M}_1| = M_0M_1' - M_1M_0' - 2\sum_{j=1}^{m}\left[F_j'X_j\sum_{i=j}^{m}F_i\right] + 2\sum_{j=1}^{m}\left[F_j'\sum_{i=j}^{m}F_iX_i\right].$$

Note the similarity of the first of these forms to formula (2.34) which is in fact a particular case of formula (3.41). Alternatively, we may obtain from formula (3.42) the particular case

$$(2.34a)\qquad |\mathfrak{M}_1| = 2\sum_{j=1}^{n}\left(F_j\sum_{i=j}^{n}F_iX_i\right) - 2\sum_{j=1}^{n}\left(F_jX_j\sum_{i=j}^{n}F_i\right)$$

which is equivalent to (2.34).

If the two distributions do not overlap, $|{}^c\mathfrak{M}_1|$ does not differ numerically from ${}^c\mathfrak{M}_1$. Let us consider the case in which there is actual overlapping, the range of non-zero frequencies extending from X_1 to X_{n+p} for D and from X_{n+1} to X_m for D'. Then formula (3.42) becomes, upon merely dropping all vanishing terms

$$(3.43)\quad \begin{aligned} |{}^c\mathfrak{M}_1| &= M_0M_1' - M_1M_0' \\ &\quad - 2\sum_{j=n+1}^{n+p}\left[F_j'X_j\sum_{i=n+1}^{j-1}F_i\right] + 2\sum_{j=n+1}^{n+p}\left[F_j'\sum_{i=j}^{n+p}F_iX_i\right]. \end{aligned}$$

On the other hand, formula (3.41) reduces, under the same circumstances, to a much less simple expression, which upon making the substitutions (2.33) and simplifying reduces to

$$(3.44)\quad \begin{aligned} |{}^c\mathfrak{M}_1| &= M_0M_1' - M_1M_0' + 2\sum_{j=n+1}^{n+p}\left[F_j'X_j\sum_{i=n+1}^{j-1}F_i\right] \\ &\quad - 2\sum_{j=n+1}^{n+p}\left[F_j'\sum_{i=j}^{n+p}F_iX_i\right] \\ &\quad - 2\sum_{j=n+1}^{n+p}F_j'X_j\sum_{i=n+1}^{n+p}F_i + 2\sum_{j=n+1}^{n+p}F_j'\sum_{i=n+1}^{n+p}F_iX_i\,. \end{aligned}$$

This result may be arrived at somewhat more easily by merely making the substitutions (2.33) directly in formula (3.43). It may be noted that formula (3.44) at once reduces to the form (2.34) if the two distributions are identical, since the additional terms all cancel. It is, however a less satisfactory result than formula (3.43) because of the larger number of terms it contains. In order to obtain a formula which resembles (2.34) more closely, we may reverse the

order of summation in formula (3.43). Observing that the terms for $j = i$ collectively vanish, we see that

$$|{}^c\mathfrak{M}_1| = M_0 M_1' - M_1 M_0' \tag{3.45}$$
$$- 2 \sum_{i=n+1}^{n+p} \left[F_i \sum_{j=n+1}^{i-1} F_j' X_j \right] + 2 \sum_{i=n+1}^{n+p} \left[F_i X_i \sum_{j=n+1}^{i-1} F_j' \right].$$

It will be seen that the simple method of numerical computation described in section 2.8 is immediately applicable to all the formulas (3.41) to (3.45). Dividing any of these expressions by ${}^c\mathfrak{M}_0$ gives $|{}^c m_1|$. For example, if formula (3.43) is used, we have

$$|{}^c m_1| = A' - A \tag{3.46}$$
$$- \frac{2}{M_0 M_0'} \left\{ \sum_{j=n+1}^{n+p} \left[F_j' X_j \sum_{i=j}^{n+p} F_i \right] - \sum_{i=j}^{n+p} \left[F_j' \sum_{i=j}^{n+p} F_i X_i \right] \right\}.$$

Substituting this value in equation (3.34), we have

$$S_1 = \frac{m_1 m_1'}{|{}^c m_1|^2} \tag{3.47}$$

a quantity which we shall call the "mean coefficient of similarity."

We now observe that S_1 is a general measure of similarity whose magnitude is affected by differences in either form or position. It may, however, be desirable to eliminate the position element, in order to isolate the form aspect. To do this it will suffice to relate the value which $|{}^c m_1|$ would have for $A = A'$, to the product $m_1 m_1'$. This value of $|{}^c m_1|$ is, in fact, its minimum; denoting it by ${}^c\mu_1$ we obtain the index

$$\mathfrak{S}_1 = \frac{m_1 m_1'}{{}^c\mu_1^2} \tag{3.48}$$

which we shall call the mean similarity ratio.

It is clear that all the above mentioned indices measure overlapping as well as similarity. Overlapping between two distributions will be greatest when their similarity is greatest, or when $|{}^c m_1|$ is a minimum. In order to bring out more clearly the overlapping aspect we may follow Gini's procedure of contrasting the actual value of a measure with its maximum value. As already pointed out, if the form of the two distributions is held constant, but their relative position is varied, the degree of overlapping, as measured by the mean similarity ratio, is greatest when the arithmetic means coincide. This method of procedure is embodied in the index

$$\mathfrak{T}_1 = \frac{{}^c\mu_1}{{}^c m_1} \tag{3.49}$$

which we shall call the "intensity of transvariation or overlapping." To calculate ${}^c\mu_1$ we may, for example, merely add the difference $A' - A = c$ to the X

values, in order to move D along the X-axis a distance of c, and then proceed to calculate $|{}^c m_1|$ in the usual manner from the adjusted X values.

3.5. If, in (3.11), $r = 2$, we have

$$M_2(D, X_j) = F'_j \sum_{i=1}^{n} (X_i - X_j)^2 F_i$$
$$= F'_j M_2 - 2X'_j F'_j M_1 + X'^2_j F'_j M_0 .$$

Summing for j then gives

$$ {}^c\mathfrak{M}_2 = M'_0 M_2 - 2M'_1 M_1 + M'_2 M_0 . \tag{3.51}$$

If we define the second aggregate unit moment as

$${}^c m_2 = \frac{{}^c\mathfrak{M}_2}{{}^c\mathfrak{M}_0}$$

then

$$\begin{aligned} {}^c m_2 &= \frac{M_2}{M_0} - 2\frac{M_1}{M_0}\frac{M'_1}{M_0} + \frac{M'_2}{M_0} \\ &= \sigma^2 + \sigma'^2 + (A - A')^2, \end{aligned} \tag{3.52}$$

where the σ and the A stand for the standard deviations and the arithmetic means of the respective distributions. Now we define the "mean square coefficient of similarity" as the value of

$$\begin{aligned} S_2 &= \frac{m_2 m'_2}{{}^c m_2^2} \\ &= \frac{4\sigma^2\sigma'^2}{[\sigma^2 + \sigma'^2 + (A - A')^2]^2}. \end{aligned} \tag{3.53}$$

It is obvious that a minimum value of S_2 requires that $A = A'$ as a necessary condition for the maximum degree of overlapping. Maximum similarity requires, in addition, $\sigma = \sigma'$, in which case $S_2 = 1$.

For a measure of similarity which is independent of difference in position between the two distributions, we define.

$$\mathfrak{S}_2 = \frac{m_2 m'_2}{{}^c\mu_2^2}, \tag{3.54}$$

where ${}^c\mu_2$ is the minimum value of ${}^c m_2$ for all positions of the two distributions, without changing their form. This is obtained by merely taking

$${}^c\mu_2 = \sigma^2 + \sigma'^2. \tag{3.55}$$

For a measure of overlapping we can follow Gini in contrasting the actual

value of ${}^c m_2$ with its minimum ${}^c\mu_2$, since the maximum of overlapping corresponds to the minimum value of ${}^c m_2$. We thus see

$$\mathfrak{T}_2 = \frac{{}^c\mu_2}{{}^c m_2} = \frac{\sigma^2 + \sigma'^2}{\sigma^2 + \sigma'^2 + (A - A')^2} \tag{3.56}$$

a measure which we shall call the "density of overlapping". Its maximum value is unity.

It may be remarked that all the indices proposed in this paragraph are easier to calculate than those of paragraph 3.4. The individual terms are all functions of only one of the two distributions; yet the resulting indices are independent of the origin chosen, and therefore free from any criticism based on doubt as to the representativeness of the arithmetic mean, in cases of marked skewness.

4. Positive and negative moments, and moments of transvariation.

4.1. The aggregate sliding total moment of two frequency distributions D and D' may be expressed in the form

$$M_r(D, X_j') = F_j' \sum_{i=1}^{j-1} (X_i - X_j)^r F_i + F_j' \sum_{i=j+1}^{m} (X_i - X_j)^r F_i \tag{4.11}$$

when both distributions have been artificially extended, if necessary, to cover the same total range, as previously described in section 3.4. We shall characterize the second term in the right member of (4.11) as the positive sliding moment, and the absolute value of the first term as the negative sliding moment. We shall denote these moments by ${}^+M_r(D, X_j)$ and ${}^-M_r(D, X_j)$. The complete moments obtained by summing these separate terms over the range of values of j we shall call the positive and negative aggregate complete moments. Thus the positive complete moment is

$$^{+c}\mathfrak{M}_r = \sum_{j=1}^{m} \left[F_j' \sum_{i=j+1}^{m} (X_i - X_j)^r F_i \right] \tag{4.12}$$

and the negative complete moment is

$$^{-c}\mathfrak{M}_r = \sum_{j=1}^{m} \left[F_j' \sum_{i=1}^{j-1} (X_j - X_i)^r F_i \right]. \tag{4.13}$$

That one of these two partial moments which is obtained from differences $X_i - X_j'$ having the opposite sense to that of the difference $\bar{X} - \bar{X}'$ will be called the moment of transvariation of the two distributions and will be denoted by the symbol ${}^T\mathfrak{M}_r$. Here, as in section 3.4, $\bar{X}$ and $\bar{X}'$ denote averages of any previously selected type. For example, if the arithmetic means are the averages selected, and if $A - A'$ is positive, then the negative aggregate moment is the moment of transvariation, and vice-versa.

In the trivial case in which the two distributions are identical, the positive and negative complete moments are equal, and both reduce to merely one half

the aggregate complete moment (computed by the use of absolute values in the case of moments of odd order).

The unit moment of transvariation will be defined as

$$(4.14) \qquad {}^{T}m_r = \frac{{}^{T}\mathfrak{M}_r}{{}^{T}\mathfrak{M}_0}$$

4.2. It is evident that the moments of transvariation can be considered as measures of overlapping. Any such moment equals zero when there is no overlapping and becomes greatest when the two distributions coincide. Taking unity to represent the maximum and zero the minimum of overlapping, we may choose as a general measure of overlapping,

$$(4.21) \qquad T_r = \frac{4\,{}^{T}m_r^2}{|m_r|\,|m_r'|} = \frac{4\,{}^{T}\mathfrak{M}_r^2}{|\mathfrak{M}_r|\,|\mathfrak{M}_r'|}.$$

It will be seen that this quantity always equals zero when there is no overlapping, and equals unity when there is complete overlapping: that is when the two distributions are identical.

5. Need for further developments. All of the measures above described were defined for the case of finite sets of magnitudes, expressed as frequency distributions D and D'. Now these sets of magnitudes may be thought of as samples drawn out of their corresponding universes. The consideration of these universes would lead to more general representations under the form of frequency functions, and the above measures would be expressed as definite integrals rather than summations. This draws attention to the need for tests of significance of the magnitude of all the above measures, especially those of overlapping, in order to allow for sampling fluctuation. Obviously, when the frequency functions are of the asymptotic type some amount of overlapping will always exist.

ON A PROBLEM OF ESTIMATION OCCURRING IN PUBLIC OPINION POLLS

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To arrive at an estimate of the number of electoral votes that will be cast for a presidential candidate a poll is taken of $\lambda_i N$ interviews in the ith state ($i = 1, \cdots, 48$) where the λ_i are fixed constants > 0 such that $\Sigma\lambda_i = 1$ and the respondent is asked for which candidate he intends to cast his vote. To estimate the number of electoral votes which candidate A will receive, the electoral votes of all the states in which the poll shows a majority for candidate A are added and their sum is used as an estimate for the number of electoral votes which candidate A will receive. In this paper certain properties of this estimate will be discussed. It will be shown that it is a biased but consistent estimate and an upper bound for the bias will be derived. Finally we shall derive that distribution of interviews which minimizes the variance of our estimate.

In all that follows we shall consider the poll as a random or stratified random sample and shall disregard the bias introduced by inaccurate answers. Our results however remain valid as long as the sampling variance is proportional to $\dfrac{1}{\sqrt{N}}$.

We shall use the following notation:

π_i = proportion of voters in the ith state who intend to vote for candidate A.

$$\epsilon_i = 1 \quad \text{if } \pi_i > \tfrac{1}{2}$$
$$0 \quad \text{if } \pi_i < \tfrac{1}{2}$$

w_i = number of electoral votes of the ith state.

p_i, e_i = sample values of π_i and ϵ_i resp.

We shall further exclude the case $\pi_i = \frac{1}{2}$.

The number of electoral votes for candidate A is then given by

$$\sum_{i=1}^{i=48} \epsilon_i w_i = \Gamma.$$

As an estimate of Γ we use the quantity

$$\sum_{i=1}^{i=48} e_i w_i = G. \tag{1}$$

Let ρ_i be the probability that $p_i > \frac{1}{2}$ and hence $e_i = 1$. Let $\lambda_i N = N_i$ be the number of interviews in the ith state. If N_i is not too small then ρ_i is given by

$$\rho_i = \int_{\frac{1}{2}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_i} e^{-(x-\pi_i)^2/2\sigma_i^2}\, dx. \tag{2}$$

In this formula $\sigma_i = \sqrt{\frac{\pi_i(1-\pi_i)}{N_i}}$ if the sample is an unstratified random sample and may be somewhat less if the sample is a stratified random sample.[1] For our purposes it is sufficient to assume that σ_i is proportional to $\frac{1}{\sqrt{N_i}}$.

We then have $E(e_i) = \rho_i$ and

$$E(G) = E(\textstyle\sum_{i=1}^{i=48} e_i w_i) = \sum_{i=1}^{i=48} \rho_i w_i . \tag{3}$$

Hence G is a biased estimate of Γ. On the other hand[2] $\underset{N\to\infty}{\text{plim}}\, p_i = \pi_i$ and hence $\underset{N\to\infty}{\text{plim}}\, e_i = \epsilon_i$ and therefore $\underset{N\to\infty}{\text{plim}}\, G = \Gamma$. That is to say G is a consistent estimate of Γ.

According to (3) the bias is given by

$$B(N) = \textstyle\sum_{i=1}^{i=48} \epsilon_i w_i - \sum_{i=1}^{i=48} \rho_i w_i = \sum_{i=1}^{i=48} (\epsilon_i - \rho_i) w_i . \tag{4}$$

We have

$$\epsilon_i - \rho_i = -\frac{1}{\sqrt{2\pi}} \int_{(\frac{1}{2}-\pi_i)/\sigma_i}^{\infty} e^{-\frac{1}{2}x^2} dx \quad \text{if} \quad \pi_i < \tfrac{1}{2}$$

$$\epsilon_i - \rho_i = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(\frac{1}{2}-\pi_i)/\sigma_i} e^{-\frac{1}{2}x^2} dx \quad \text{if} \quad \pi_i > \tfrac{1}{2}.$$

For a stratified as well as for an unstratified sample σ_i is proportional to $\frac{1}{\sqrt{N_i}}$ and we therefore put

$$\frac{\frac{1}{2} - \pi_i}{\sigma_i} = \begin{cases} \gamma_i\sqrt{N_i} & \text{if} \quad \pi_i < \frac{1}{2} \\ -\gamma_i\sqrt{N_i} & \text{if} \quad \pi_i > \frac{1}{2} \end{cases}. \tag{5}$$

Then we have in both cases

$$|\epsilon_i - \rho_i| = \frac{1}{\sqrt{2\pi}} \int_{\gamma_i\sqrt{N_i}}^{\infty} e^{-\frac{1}{2}x^2} dx. \tag{6}$$

We have for $a > 0$

$$\begin{aligned} \int_a^{\infty} e^{-\frac{1}{2}x^2} dx &\leq h(e^{-\frac{1}{2}a^2} + e^{-\frac{1}{2}(a+h)^2} + e^{-\frac{1}{2}(a+2h)^2} + \cdots) \\ &< e^{-\frac{1}{2}a^2} h(1 + e^{-ah} + e^{-2ah} + \cdots) \\ &= e^{-\frac{1}{2}a^2} \cdot \frac{h}{1 - e^{-ah}} \end{aligned}$$

for every value h.

Since $\lim_{h\to 0} \frac{h}{1 - e^{-ah}} = \frac{1}{a}$ we have

$$\int_a^{\infty} e^{-\frac{1}{2}x^2} dx \leq \frac{e^{-\frac{1}{2}a^2}}{a} \quad \text{for every } a > 0. \tag{7}$$

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[1] The variance in public opinion polls is somewhat larger than the random sampling variance due to the fact that a cluster sample is used and not a random sample. For the same reason the estimate p_i of π_i may be biased.

[2] For the notation used here see: H. B MANN AND A. WALD, "On stochastic limit and order relationships". *Annals of Math. Stat.*, (1943), pp. 217–227.

From (6) and (7) we obtain

$$|\epsilon_i - \rho_i| \leq \frac{e^{-\frac{1}{2}\gamma_i^2 N_i}}{\sqrt{2\pi N_i}\,\gamma_i}. \tag{8}$$

From (4) and (8) we have

$$|B(N)| \leq \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{i=48} w_i \frac{e^{-\frac{1}{2}\gamma_i^2 N_i}}{\gamma_i \sqrt{N_i}}. \tag{9}$$

Formula (9) is valid whenever $\pi_i \neq \frac{1}{2}$ and shows that $B(N)$ converges rapidly to 0 for all values $\pi_i \neq \frac{1}{2}$.

To obtain an approximate idea of the magnitude of the bias we may in (4) replace ϵ_i and ρ_i by their sample values e_i and r_i. The quantity $\sum_{i=1}^{i=48} w_i (e_i - r_i)$ can, however, not be regarded as an estimate of $B(N)$.

We now proceed to compute the standard error of G. We may consider the poll as 48 single experiments where the probability of success in the ith experiment is given by ρ_i where

$$\frac{1}{\sqrt{2\pi}} \int_{\gamma_i\sqrt{N_i}}^{\infty} e^{-\frac{1}{2}x^2}\,dx = \begin{cases} \rho_i & \text{if} \quad \pi_i < \frac{1}{2} \\ 1 - \rho_i & \text{if} \quad \pi_i > \frac{1}{2} \end{cases}.$$

Hence the variance of G is given by

$$\sigma^2 = \sum_{i=1}^{i=48} \rho_i (1 - \rho_i) w_i^2. \tag{10}$$

As an estimate of σ^2 we can use the quantity S^2 obtained by replacing ρ_i by its sample value.

We shall consider that distribution of interviews as best which minimizes $E[(G - \Gamma)^2]$.

We have

$$E[(G - \Gamma)^2] = \sigma^2 + B^2(N)$$

We therefore consider the problem of minimizing $\sigma^2 + B^2(N)$ under the restriction $\sum_{i=1}^{i=48} N_i = N$.

We have

$$\frac{\partial \sigma^2}{\partial N_i} = \frac{\partial \sigma^2}{\partial \rho_i} \frac{\partial \rho_i}{\partial N_i} = w_i^2 (1 - 2\rho_i) \frac{\partial \rho_i}{\partial N_i}$$

$$\frac{\partial B^2(N)}{\partial N_i} = 2B(N) \frac{\partial B(N)}{\partial N_i} = -2w_i B(N) \frac{\partial \rho_i}{\partial N_i}$$

$$\frac{\partial \rho_i}{\partial N_i} = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\gamma_i^2 N_i} \frac{\gamma_i}{2\sqrt{N_i}} \quad \text{if} \quad \pi_i < \tfrac{1}{2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\gamma_i^2 N_i} \frac{\gamma_i}{2\sqrt{N_i}} \quad \text{if} \quad \pi_i > \tfrac{1}{2}.$$

Hence applying the method of Lagrange operators, we obtain

$$\frac{\partial[\sigma^2 + B^2(N)]}{\partial N_i} = \frac{\partial \rho_i}{\partial N_i} w_i[w_i(1 - 2\rho_i) - 2B(N)] = \lambda, \quad i = 1 \cdots 48. \tag{11}$$

$$\sum_{i=1}^{i=48} N_i = N.$$

The parameters γ_i and π_i in equation (11) can be estimated from a previous poll.[3] It is not certain that (11) has always solutions. However if the quantity $\sigma^2 + B^2(N)$ has a minimum for a set of values $N_1, \cdots, N_{48}$ with $N_i \neq 0$ ($i = 1, \cdots, 48$) then (11) must have a solution

One might be induced to try to estimate $\Sigma \rho_i w_i$ directly by using $r_i = \frac{1}{\sqrt{2\pi}} \int_{(\frac{1}{2}-p_i)/s_i}^{\infty} e^{-x^2/2}\, dx$ as an estimate of ρ_i. It is easy to see that r_i is a consistent estimate of ϵ_i. It will be shown however that this estimate is more biased than the estimate (1).

Since σ_i differs only very little from its sample estimate s_i we may replace this sample estimate by σ_i. We then have

$$\begin{aligned} E(r_i) &= E\left(\frac{1}{\sqrt{2\pi}\,\sigma_i} \int_{\frac{1}{2}}^{\infty} e^{-(x-p_i)^2/2\sigma_i^2}\, dx\right) \\ &= \frac{1}{2\pi\sigma_i^2} \int_{-\infty}^{+\infty} \left(\int_{\frac{1}{2}}^{\infty} e^{-(x-p_i)^2/2\sigma_i^2}\, dx\right) e^{-(p_i-\pi_i)^2/2\sigma_i^2}\, dp_i \\ &= \frac{1}{2\pi\sigma_i^2} \int_{-\infty}^{+\infty} \int_{\frac{1}{2}}^{\infty} e^{-[(x-p_i)^2+(p_i-\pi_i)^2]/2\sigma_i^2}\, dx\, dp_i . \end{aligned}$$

Now

$$(x - p_i)^2 + (p_i - \pi_i)^2 = \frac{(x - \pi_i)^2}{2} + 2\left(p_i - \frac{\pi_i + x}{2}\right)^2.$$

Hence

$$E(r_i) = \frac{1}{2\pi\sigma_i^2} \int_{\frac{1}{2}}^{\infty} e^{-(x-\pi_i)^2/4\sigma_i^2} \left(\int_{-\infty}^{+\infty} e^{-(p_i-\frac{1}{2}(\pi_i+x))^2/\sigma_i^2}\, dp_i\right) dx.$$

The second integral is equal to $\sqrt{\pi\sigma_i^2}$. Hence

$$E(r_i) = \frac{1}{2\sqrt{\pi\sigma_i^2}} \int_{\frac{1}{2}}^{\infty} e^{-(x-\pi_i)^2/4\sigma_i^2}\, dx = \frac{1}{\sqrt{2\pi}} \int_{(\frac{1}{2}-\pi_i)/\sigma_i\sqrt{2}}^{\infty} e^{-x^2/2}\, dx .$$

[3] If π_i for any i were very close to $\frac{1}{2}$ then it would be of little use to poll the ith state. Hence, in this case formula (11) gives a small value for N_i. However, the π_i are never accurately known. The following procedure might be recommended for determining the best distribution of interviews: If for one particular i the sample value of π_i as estimated from a previous poll is too close to $\frac{1}{2}$ determine, using the N_i of the previous poll, that value $\bar{\pi}_i$ of π_i for which the probability is $\frac{1}{10}$ that p_i is larger than $\frac{1}{2}$ and substitute in (11) $\bar{\pi}_i$ for π_i. In all other cases substitute the sample value.

If several polls are taken it is advisable to use all of them but the last one to estimate as closely as possible the values of the π_i. The sample of the last poll before the election should be distributed according to (11).

From (12) we see that $E(r_i) < \rho_i$ if $\pi_i > \frac{1}{2}$ and $E(r_i) > \rho_i$ if $\pi_i < \frac{1}{2}$. Thus in every case this estimate is more biased than the estimate (1).

On the other hand, we shall now show that $E[(\epsilon_i - r_i)^2]$ is always smaller than $E[(\epsilon_i - e_i)^2]$. Since $\epsilon_i = 1$ if $\pi_i > \frac{1}{2}$ and $\epsilon_i = 0$ if $\pi_i < \frac{1}{2}$ it is easy to verify that $E[(\epsilon_i - r_i)^2]$ has the same value for $\pi_i = a$ as for $\pi_i = 1 - a$ and the same is true for $E[(\epsilon_i - e_i)^2]$. We may, therefore, without loss of generality assume that $\pi_i < \frac{1}{2}$.

Thus we have to show that

$$E(r_i^2) \le E(e_i^2) = \rho_i = \int_{(\frac{1}{2}-\pi_i)/\sigma_i}^{\infty} e^{-\frac{1}{2}x^2}\,dx \qquad \text{if } \pi_i < \tfrac{1}{2}. \tag{13}$$

We have

$$E(r_i^2) = \frac{1}{\sqrt{2\pi}\sigma_i}\int_{-\infty}^{+\infty} e^{-(p_i-\pi_i)^2/2\sigma_i^2}\left(\int_{(\frac{1}{2}-p_i)/\sigma_i}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\,dx\right)^2 dp_i$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_{\frac{1}{2}}^{\infty}\int_{\frac{1}{2}}^{\infty}\frac{1}{2\pi\sigma_i^3}e^{-(1/2\sigma_i^2)Q(x,y,p_i)}\,dx\,dy\,dp_i\,.$$

Now

$$Q(x, y, p_i) = (x - p_i)^2 + (y - p_i)^2 + (p_i - \pi_i)^2$$
$$= 3\left(p_i - \frac{x + y + \pi_i}{3}\right)^2 + \frac{1}{6}(x + y - 2\pi_i)^2 + \frac{1}{2}(x - y)^2.$$

Putting

$$p_i' = \frac{\sqrt{3}\left(p_i - \dfrac{x + y + \pi_i}{3}\right)}{\sigma_i}, \quad x' = \frac{1}{\sqrt{6}}\frac{(x + y - 2\pi_i)}{\sigma_i},$$
$$y' = \frac{1}{\sqrt{2}}\frac{(x - y)}{\sigma_i}, \quad \frac{1 - 2\pi_i}{\sqrt{6}\sigma_i} = a,$$

we obtain

$$E(r_i^2) = \frac{1}{(\sqrt{2\pi})^3}\int_{-\infty}^{+\infty}\int_a^{\infty} e^{-\frac{1}{2}p'^2}e^{-\frac{1}{2}x'^2}\left(\int_{\sqrt{3}(a-x')}^{\sqrt{3}(x'-a)} e^{-\frac{1}{2}y'^2/2}\,dy'\right)dx'\,dp'$$
$$= \frac{1}{2\pi}\int_a^{\infty} e^{-\frac{1}{2}x^2}\int_{\sqrt{3}(a-x)}^{\sqrt{3}(x-a)} e^{-y^2}\,dy\,dx.$$

Now for $\pi_i = \frac{1}{2}$ we have $a = 0$, and for $\pi_i < \frac{1}{2}$ we have $a > 0$. For $a = 0$ we obviously have $E(r_i^2 \le E(e_i^2)$. Further $\lim\limits_{a\to\infty} E(r_i^2) = \lim\limits_{a\to\infty} E(e_i^2) = 0$ hence (13) is proved if we can show that

$$F(a) = E(r_i^2) - E(e_i^2) = \frac{1}{2\pi}\int_a^{\infty} e^{-\frac{1}{2}x^2}\int_{\sqrt{3}(a-x)}^{\sqrt{3}(x-a)} e^{-\frac{1}{2}y^2}\,dy\,dx - \frac{1}{\sqrt{2\pi}}\int_{\sqrt{\frac{3}{2}}a}^{\infty} e^{-\frac{1}{2}x^2}\,dx$$

is a monotonically increasing function of a. Differentiating $F(a)$ with respect to a we obtain

$$\begin{aligned} \frac{dF(a)}{da} &= -\frac{\sqrt{3}}{\pi}\int_a^\infty e^{-\frac{1}{4}(4x^2-6ax+3a^2)} + \frac{\sqrt{3}}{2\sqrt{\pi}}e^{-(3/4)a^2} \\ &= \frac{-\sqrt{3}}{\pi}e^{-(3/4)a^2}\int_a^\infty e^{-4(x-(3/4)a)^2}\,dx + \frac{\sqrt{3}}{2\sqrt{\pi}}e^{-(3/4)a^2} \\ &= \frac{-\sqrt{3}}{2\pi}e^{-(3/4)a^2}\int_{\frac{a}{2}}^\infty e^{-\frac{1}{2}x^2}\,dx + \frac{\sqrt{3}}{2\sqrt{\pi}}e^{-(3/4)a^2}. \end{aligned} \tag{14}$$

Hence for $a \geq 0$ we have

$$\frac{dF}{da} \geq \frac{-\sqrt{3}}{2\sqrt{2\pi}}e^{-\frac{3}{4}a^2} + \frac{\sqrt{3}}{2\sqrt{\pi}}e^{-\frac{3}{4}a^2} \geq 0.$$

Hence we have proved

$$\begin{gathered} E[(\epsilon_i - r_i)^2] = \frac{1}{2\pi}\int_{|a|}^\infty e^{-\frac{1}{2}x^2}\int_{\sqrt{3}(|a|-x)}^{\sqrt{3}(x-|a|)} e^{-y^2}\,dy\,dx \leq E[(\epsilon_i - e_i)^2], \\ a = \frac{1-2\pi_i}{\sqrt{6}\,\sigma_i}. \end{gathered} \tag{15}$$

Since

$$E[(\epsilon_i - e_i)^2] - E[(\epsilon_i - r_i)^2]$$

is largest when $\pi_i = \frac{1}{2}$ we also have

$$E[(\epsilon_i - r_i)^2] \geq |\epsilon_i - \rho_i| - \left[\frac{1}{2} - \frac{1}{2\pi}\int_0^\infty e^{-\frac{1}{2}x^2}\int_{-\sqrt{3}x}^{+\sqrt{3}x} e^{-\frac{1}{2}y^2}\,dy\,dx\right]$$

or

$$|\epsilon_i - \rho_i| \geq E[(\epsilon_i - r_i)^2] \geq \frac{1}{2\pi}\int_0^\infty e^{-\frac{1}{2}x^2}\int_{-\sqrt{3}x}^{+\sqrt{3}x} e^{-\frac{1}{2}y^2}\,dy\,dx - |\tfrac{1}{2} - \rho_i|. \tag{16}$$

Because of (15), r_i although more biased may in many cases be preferable to e_i as an estimate of ϵ_i.

NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

A COMBINATORIAL FORMULA AND ITS APPLICATION TO THE THEORY OF PROBABILITY OF ARBITRARY EVENTS[1]

BY KAI-LAI CHUNG AND LIETZ C HSU

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An important principle, known as a proposition in formal logic or the method of cross-classification can be stated as follows.[1]

Let F and f be any two functions of combinations out of $(\nu) = (1, 2, \cdots, n)$. Then the two formulas

$$F((\alpha)) = \sum_{(\beta) \in (\nu)-(\alpha)} f((\alpha) + (\beta)) \tag{1.1}$$

$$f((\alpha)) = \sum_{(\beta) \in (\nu)-(\alpha)} (-1)^b F((\alpha) + (\beta)) \tag{2.1}$$

are equivalent.

As an immediate application to the theory of probability of arbitrary events, we have the set of inversion formulas[2]

$$p((\alpha)) = \sum_{(\beta) \in (\nu)-(\alpha)} p[(\alpha) + (\beta)] \tag{3.1}$$

$$p[(\alpha)] = \sum_{(\beta) \in (\nu)-(\alpha)} (-1)^b p((\alpha) + (\beta)) \tag{4.1}$$

where $p((\alpha))$ is the probability of the occurrence of at least $E_{\alpha_1}, E_{\alpha_2}, \cdots, E_{\alpha_a}$ out of n arbitrary events $E_1, E_2, \cdots, E_n$ and $p[(\alpha)]$ is the probability of the occurrence of $E\alpha_1, E\alpha_2, \cdots, E\alpha_a$ and no others among the n events, $(\alpha_1, \alpha_2, \cdots, \alpha_a)$ denoting a combination of the integers $(1, 2, \cdots, n)$. They can be made to play a central rôle in the theory, since they supply a method for converting the fundamental systems of probabilities, $p[(\alpha)]$ and $p((\alpha))$, one into the other.

We may further generalize (1.1) and (2.1) by considering combinations with repetitions. Let such a combination be written as

$$(\alpha) = (\alpha^r) = (\alpha_1^{r_1}\alpha_2^{r_2} \cdots \alpha_a^{r_a})$$

[1] For the notations and definitions see K. L. CHUNG, "On fundamental systems of probabilities of a finite number of events," *Annals of Math. Stat.*, Vol. 14 (1943), pp. 123–133.

[2] Cf. FRÉCHET, *Les probabilités associées à un système d'événements compatibles et dépendants*, Hermann, Paris (1939), formulas (55) and (58).

where r_i ($r_i \geq 1$) denotes the number of repetitions of the number α_i, $i = 1, 2, \cdots, a$. Correspondingly we write

$$(\alpha)' = (\alpha_1 \alpha_2 \cdots \alpha_a)$$

and call it the reduced combination corresponding to (α).

If there are n distinct elements $(1, 2, \cdots, n)$ in question, we may write every combination in the form

$$(1^{r_1} 2^{r_2} \cdots n^{r_n})$$

where each r_i is zero or a positive integer. We say that $(1^{s_1} 2^{s_2} \cdots n^{s_n})$ belongs to $(1^{r_1} 2^{r_2} \cdots n^{r_n})$ and write

$$(1^{s_1} 2^{s_2} \cdots n^{s_n}) \in (1^{r_1} 2^{r_2} \cdots n^{r_n})$$

if and only if for each i, $i = 1, 2, \cdots, n$, we have $s_i \leq r_i$. We write

$$(1^{r_1} 2^{r_2} \cdots n^{r_n}) + (1^{s_1} 2^{s_2} \cdots n^{s_n}) = (1^{r_1+s_1} 2^{r_2+s_2} \cdots n^{r_n+s_n});$$

and if $(1^{s_1} 2^{s_2} \cdots n^{s_n}) \in (1^{r_1} 2^{r_2} \cdots n^{r_n})$,

$$(1^{r_1} 2^{r_2} \cdots n^{r_n}) - (1^{s_1} 2^{s_2} \cdots n^{s_n}) = (1^{r_1-s_1} 2^{r_2-s_2} \cdots n^{r_n-s_n}).$$

We define a generalized Möbius function $\mu((\alpha))$ for combinations (with or without repetitions) as follows

$$\mu((\alpha)) = \begin{matrix} (-1)^a & \text{if} & (\alpha) = (\alpha)' \\ 0 & \text{if} & (\alpha) \neq (\alpha)'. \end{matrix}$$

This function has the property

$$\sum_{(\beta) \in (\alpha)} \mu((\beta)) = \begin{matrix} 1 & \text{if} & (\alpha) = (0) \\ 0 & \text{if} & (\alpha) \neq (0). \end{matrix}$$

For we have

$$\sum_{(\beta) \in (\alpha)} \mu((\beta)) = \sum_{(\beta) \in (\alpha)'} (-1)^b = \sum_{b=0}^{a'} (-1)^b \binom{a'}{b}$$

$$= \begin{matrix} 1 & \text{if} & a' = 0 \\ 0 & \text{if} & a' \neq 0 \end{matrix} = \begin{matrix} 1 & \text{if} & (\alpha) = (0) \\ 0 & \text{if} & (\alpha) \neq (0). \end{matrix}$$

Now we state and prove the following general theorem.

THEOREM. *Let* $(\alpha)_i = (\alpha_{i1}^{r_{i1}} \alpha_{i2}^{r_{i2}} \cdots \alpha_{ia_i}^{r_{ia_i}})$ *and* $(\nu)_i = (1^{\lambda_{i1}} 2^{\lambda_{i2}} \cdots n_i^{\lambda_{in_i}})$ *where* λ_{ij} *and* n_i *are finite and* $1 \leq r_{ij} \leq \lambda_{ij}$, $1 \leq a_i \leq n_i$ *for* $i = 1, 2, \cdots, m$ *and* $j = 1, 2, \cdots, n_i$. *Then for any two functions of the m combinations (with repetitions),* $(\alpha)_1, (\alpha)_2, \cdots, (\alpha)_m$ *out of* $(\nu)_1, (\nu)_2, \cdots, (\nu)_m$, *the two sets of formulas:*

$$\text{(1)} \quad \begin{aligned} &F((\alpha)_1, (\alpha)_2, \cdots (\alpha)_m) \\ &\quad = \sum_{(\beta)_i \in (\nu)_i - (\alpha)_i} f((\alpha)_1 + (\beta)_1, (\alpha)_2 + (\beta)_2, \cdots, (\alpha)_m + (\beta)_m) \end{aligned}$$

and

$$
(2) \qquad \begin{aligned} & f((\alpha)_1, (\alpha)_2, \cdots, (\alpha)_m) \\ & = \sum_{(\beta)_i \in (\nu)_i - (\alpha)_i} \left[\prod_{i=1}^{m} \mu((\beta)_i) \right] F((\alpha)_1 + (\beta)_1, (\alpha)_2 + (\beta)_2, \cdots, (\alpha)_m + (\beta)_m) \end{aligned}
$$

are equivalent.

PROOF. To deduce (2) from (1)

$$
\begin{aligned}
\sum_{(\beta)_i \in (\nu)_i - (\alpha)_i} & \left[\prod_{i=1}^{m} \mu((\beta)_i) \right] F((\alpha)_1 + (\beta)_1, \cdots, (\alpha)_m + (\beta)_m) \\
&= \sum_{(\beta)_i \in (\nu)_i - (\alpha)_i} \left[\prod_{i=1}^{m} \mu((\beta)_i) \right] \sum_{(\gamma)_i \in (\nu)_i - (\alpha)_i - (\beta)_i} \\
&\qquad \cdot f((\alpha)_1 + (\beta)_1 + (\gamma)_1, \cdots, (\alpha)_m + (\beta)_m + (\gamma)_m) \\
&= \sum_{(\delta)_i \in (\nu)_i - (\alpha)_i} f((\alpha)_1 + (\delta)_1, \cdots, (\alpha)_m + (\delta)_m) \\
&\qquad \cdot \sum_{(\gamma)_i \in (\delta)_i} \prod_{i=1}^{m} \mu((\delta)_i - (\gamma)_i).
\end{aligned}
$$

Evidently we have

$$
\begin{aligned}
\sum_{(\gamma)_i \in (\delta)_i} \prod_{i=1}^{m} \mu((\delta)_i - (\gamma)_i) &= \prod_{i=1}^{m} \left\{ \sum_{(\gamma)_i \in (\delta)_i} \mu((\delta)_i - (\gamma)_i) \right\} \\
&= \prod_{i=1}^{m} \left\{ \sum_{(\gamma)_i \in (\delta)_i} \mu((\gamma)_i) \right\} = \begin{matrix} 1 & \text{if} \quad (\delta)_i = (0) \quad \text{for} \quad i = 1, \cdots, m \\ 0 & \text{otherwise} \end{matrix}
\end{aligned}
$$

by the property of the μ-function. Hence the preceding sum reduces to $f((\alpha)_1, \cdots, (\alpha)_m)$ in accord with (2).

(1) is deduced from (2) in a similar way.

Although the general case is not without importance in the treatment of several sets of events,[3] we shall for the sake of convenience restrict ourselves to the special case $m = 1$.

In order to apply these formulas we must first introduce combinations with repetitions into the theory of arbitrary events. This can be done in various ways. Firstly, we may consider the number of occurrences of each event in a given time-interval or in a series of trials not necessarily independent. Secondly, we may regard each event as possessing various degrees of intensity. If the event E_i occurs r_i times in a given time-interval or occurs with r_i degrees of intensity, we write it as $E_i^{r_i}$. Hereafter we shall make use of the first interpreta-

[3] Cf. FRÉCHET, Loc Cit pp 50–52; also, K. L. CHUNG, "Generalization of Poincaré's formula in the theory of probability," *Annals of Math. Stat.*, Vol. 14 (1943). We may note that our general theorem may be used to give another proof of the generalized Poincaré's formula for several sets of events

tion and we shall assume that the maximum number of occurrences of each event is finite:

$$0 \leqq r_i \leqq \lambda_i, \qquad i = 1, \cdots, n.$$

We define

$p[E_1^{r_1} \cdots E_n^{r_n}] = p[(\nu^r)] =$ The probability that E_i occurs exactly r_i times in the given time-interval.

$p(E_1^{r_1} \cdots E_n^{r_n}) = p((\nu^r)) =$ The probability that E_i occurs at least r_i times in the given time-interval.

These quantities play the same rôle as the $p[(\alpha)]$'s and $p((\alpha))$'s in the ordinary theory. Evidently the probability of every complex event in question can be expressed as the sum of certain $p[(\nu^r)]$'s. To prove that the $p((\nu^r))$'s also form a fundamental system of quantities we have only to express $p[(\nu^r)]$'s in terms of the $p((\nu^r))$'s. This is given immediately by an application of the general theorem with $m = 1$. For we have in an obvious way

$$p(E_1^{r_1} \cdots E_n^{r_n}) = \sum_{r_i \leqq t_i \leqq \lambda_i} p[E_1^{t_1} \cdots E_n^{t_n}]$$

or

$$p((\nu^r)) = \sum_{(\nu^s) \in (\nu^\lambda)-(\nu^r)} p[(\nu^r) + (\nu^s)] = \sum_{(\nu^s) \in (\nu^{\lambda-r})} p[(\nu^{r+s})]. \tag{3}$$

Hence we obtain the inversion

$$p[(\nu^r)] = \sum_{(\nu^s) \in (\nu^\lambda)-(\nu^r)} \mu((\nu^s))p((\nu^r) + (\nu^s)). \tag{4}$$

Let (α') denote a running combination without repetitions. Then since $\mu((\nu^s)) = 0$ unless (ν^s) is a (ν'),

$$p[(\nu^r)] = \sum_{(\alpha') \in (\nu^{\lambda-r})} \mu((\alpha'))p((\nu^r) + (\alpha')) = \sum_{(\alpha') \in (\nu^{\lambda-r})} (-1)^a p((\nu^r) + (\alpha'))^* \tag{4'}$$

The set of formulas (3) and (4) generalize (3.1) and (4.1).

Corresponding to the $p_{[a]}((\nu))$ for the ordinary events we define for $a + b + \cdots = n$ and $r, s, \cdots$ all distinct:

$p_{[a]^r,[b]^s,\cdots}(E_1^{\lambda_1} \cdots E_n^{\lambda_n}) =$ The probability that among n events $E_1, E_2, \cdots, E_n$ exactly a events occur r times, exactly b events occur s times and so on.

By (4) we easily obtain

$$p_{[a]^r,[b]^s,\cdots}((\nu^\lambda)) = \sum_S \sum_{(\nu^x) \in (\nu^\lambda)-((\alpha)^r+(\beta)^s+\cdots)} \mu((\nu^x))p((\nu^x) + (\alpha)^r + (\beta)^s + \cdots) \tag{5}$$

where $(\alpha)^r = (E_{\alpha_1}^r \cdots E_{\alpha_a}^r)$, $(\beta)^s = (E_{\beta_1}^s \cdots E_{\beta_b}^s)$, $\cdots$ and the first summation is a symmetric sum which extends to all $n!/a!b! \cdots$ different combinations $(\alpha_1 \cdots \alpha_a)$, $(\beta_1 \cdots \beta_b)$, $\cdots$ out of $(\nu) = (1, 2 \cdots n)$.

The equality (5) is obviously a generalization of Poincaré's formula.

Similarly for the probabilities in the definition of which the word "exactly"

is sometimes substituted for the words "at least." Of course we can express all of them in terms of the $p[(\nu^r)]$'s or of the $p((\nu^r))$'s. However elegant formulas such as in the ordinary theory seem to be lacking.

Finally, we may also consider conditions of existence for the $p[(\nu^r)]$'s and the $p((\nu^r))$'s For the former system the conditions are that they be all non-negative and that their sum be 1. For the latter system, the conditions are given by (4′), viz for every $(\nu^r) \epsilon (\nu^\lambda)$,

$$\sum_{(\alpha') \epsilon (\nu^{\lambda - r})} \mu((\alpha'))p((\nu^r) + (\alpha)) \geqq 0.$$

These conditions are necessary and sufficient since (3) and (4) are equivalent.

ON THE MECHANICS OF CLASSIFICATION

By Carl F. Kossack

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1. Introduction. Wald[1] has recently determined the distribution of the statistic U to be used in the classification of an observation, z_i $(i = 1, 2, \cdots, p)$, as coming from one of two populations. He also determined the critical region which is most powerful for such a classification. It is the purpose of this paper to show how such a classification statistic under the assumption of large sampling can be applied in an actual problem and to present a systematic approach to the necessary computations.

The data used in this demonstration are those which were obtained from the A.S.T.P. pre-engineering trainees assigned to the University of Oregon. The problem considered is that of classifying a trainee as to whether he will do unsatisfactory or satisfactory work[2] in the first term mathematics course (Intermediate Algebra). The variables used in the classification are: (1) A Mathematics Placement Test Score. This is the score obtained by the trainee on a fifty-minute elementary mathematics test (including elementary algebra). The test was given to each trainee on the day that he arrived on the campus. (2) A High School Mathematics Score. A trainee's high school mathematics record was made into a score by giving 1 point to students who had had no high school algebra, 2 points to students with an F in first-year, high-school algbra and no second-year algebra, 3 points for a D, $\cdots$, 10 points for an average grade of A in first- and second-year algebra. (3) The Army General Classification Test Score. An individual needed a score of 115 or better in order to be assigned to the A.S.T.P. These data were obtained for 305 trainees along with the actual

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[1] Abraham Wald, "On a statistical problem arising in the classification of an individual into one of two groups," *Annals of Math. Stat.*, Vol. 15, (1944), No. 2.

[2] Unsatisfactory work was defined as a grade of F or D in the course (failure or the lowest passing grade)

grade made by them in the algebra course. Trainees who had had college work were not included in the study.

2. Steps in the Computation of U and the Critical Region. Let

π_1 be the population of individuals who do unsatisfactory work in their first-term mathematics course.

π_2 be the population of individuals who do satisfactory work.

N_1 and N_2 = respectively the number of observed individuals in π_1 and π_2.

$x_{1\alpha}$ and $y_{1\alpha}$ = respectively the Mathematics Placement Test Score for the αth individual observed in π_1 and π_2.

$x_{2\alpha}$ and $y_{2\alpha}$ = respectively the High School Mathematics Score.

$x_{3\alpha}$ and $y_{3\alpha}$ = respectively the Army General Classification Test Score.

Step 1 *Computation of Summations*

$$
\begin{array}{rlrl}
N_1 &= 96 & N_2 &= 209 \\
\sum_\alpha x_{1\alpha} &= 3570 & \sum_\alpha y_{1\alpha} &= 11450 \\
\Sigma x_{2\alpha} &= 547 & \Sigma y_{2\alpha} &= 1567 \\
\Sigma x_{3\alpha} &= 11745 & \Sigma y_{3\alpha} &= 26684 \\
\Sigma x_{1\alpha}^2 &= 145476 & \Sigma y_{1\alpha}^2 &= 672452 \\
\Sigma x_{2\alpha}^2 &= 3509 & \Sigma y_{2\alpha}^2 &= 12577 \\
\Sigma x_{3\alpha}^2 &= 1439559 & \Sigma y_{3\alpha}^2 &= 3421996 \\
\Sigma x_{1\alpha}x_{2\alpha} &= 21012 & \Sigma y_{1\alpha}y_{2\alpha} &= 88774 \\
\Sigma x_{1\alpha}x_{3\alpha} &= 436964 & \Sigma y_{1\alpha}y_{3\alpha} &= 1469302 \\
\Sigma x_{2\alpha}x_{3\alpha} &= 66731 & \Sigma y_{2\alpha}y_{3\alpha} &= 200150 \\
\Sigma(x_{1\alpha} - \bar{x}_1)^2 &= 12716.625 & \Sigma(y_{1\alpha} - \bar{y}_1)^2 &= 45167.311 \\
\Sigma(x_{2\alpha} - \bar{x}_2)^2 &= 392.240 & \Sigma(y_{2\alpha} - \bar{y}_2)^2 &= 828.249 \\
\Sigma(x_{3\alpha} - \bar{x}_3)^2 &= 2631.656 & \Sigma(y_{3\alpha} - \bar{y}_3)^2 &= 15125.876 \\
\Sigma(x_{1\alpha} - \bar{x}_1)(x_{2\alpha} - \bar{x}_2) &= 670.438 & \Sigma(y_{1\alpha} - \bar{y}_1)(y_{2\alpha} - \bar{y}_2) &= 2926.392 \\
\Sigma(x_{1\alpha} - \bar{x}_1)(x_{3\alpha} - \bar{x}_3) &= 196.812 & \Sigma(y_{1\alpha} - \bar{y}_1)(y_{3\alpha} - \bar{y}_3) &= 7427.359 \\
\Sigma(x_{2\alpha} - \bar{x}_2)(x_{3\alpha} - \bar{x}_3) &= -191.031 & \Sigma(y_{2\alpha} - \bar{y}_2)(y_{3\alpha} - \bar{y}_3) &= 83.837
\end{array}
$$

Step 2. *Computation of Statistics.*

$$
\begin{array}{rlrl}
\bar{x}_1 &= 37.188 & \bar{y}_1 &= 54.785 \\
\bar{x}_2 &= 5.6979 & \bar{y}_2 &= 7.4976 \\
\bar{x}_3 &= 122.3438 & \bar{y}_3 &= 127.6746
\end{array}
$$

$$
s_{ij} = \frac{\Sigma(x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j) + \Sigma(y_{i\alpha} - y_i)(y_{j\alpha} - y_j)}{N_1 + N_2 - 2}
$$

$$
\begin{array}{rlrl}
s_{11} &= 191.04 & s_{12} &= 11.871 \\
s_{22} &= 4\,0280 & s_{13} &= 25.162 \\
s_{33} &= 58\,606 & s_{23} &= -.35378
\end{array}
$$

Step 3. *Computation of Inverse Matrix* $| s^{ij} |$

$$| s_{ij} | = \begin{vmatrix} 191.04 & 11.871 & 25.162 \\ 11.871 & 4.0280 & -.35378 \\ 25.162 & -.35378 & 58.606 \end{vmatrix} = 34053$$

$$\begin{aligned} s^{11} &= .0069286 & s^{12} &= -.020692 \\ s^{22} &= .31019 & s^{13} &= -.0030996 \\ s^{33} &= .018459 & s^{23} &= .010756 \end{aligned}$$

Step 4. *Computation of the Classification Equation.*

$$\begin{aligned} U = {} & [s^{11}(\bar{y}_1 - \bar{x}_1) + s^{12}(\bar{y}_2 - \bar{x}_2) + s^{13}(\bar{y}_3 - \bar{x}_3)]\cdot z_1 \\ & + [s^{21}(\bar{y}_1 - \bar{x}_1) + s^{22}(\bar{y}_2 - \bar{x}_2) + s^{23}(\bar{y}_3 - \bar{x}_3)]\cdot z_2 \\ & + [s^{31}(\bar{y}_1 - \bar{x}_1) + s^{32}(\bar{y}_2 - \bar{x}_2) + s^{33}(\bar{y}_3 - \bar{x}_3)]\cdot z_3 \end{aligned}$$

where z_i plays the same role for individuals to be classified as $x_{i\alpha}$ and $y_{i\alpha}$ do for observed individuals.

$$U = .068160\, z_1 + .25147\, z_2 + .063215\, z_3$$

Step 5. *Computation of the Critical Region (assuming* $W_1 = W_2$*)*

$$\bar{\alpha}_1 = .068160\, \bar{x}_1 + .25147\, \bar{x}_2 + .063215\, \bar{x}_3 = 11.702$$

$$\bar{\alpha}_2 = .068160\, \bar{y}_1 + .25147\, \bar{y}_2 + .063215\, \bar{y}_3 = 13.691$$

$$\tfrac{1}{2}(\bar{\alpha}_1 + \bar{\alpha}_2) = 12.696$$

Therefore,

For $U \leq 12.696$ classify the individual as coming from π_1 population.
For $U > 12.696$ classify the individual as coming from π_2 population.

Step 6. *Computation of the Efficiency of Classification.*

$$\begin{aligned} \bar{\sigma}^2 = {} & s^{11}(\bar{y}_1 - \bar{x}_1)(\bar{y}_1 - \bar{x}_1) + s^{12}(\bar{y}_1 - \bar{x}_1)(\bar{y}_2 - \bar{x}_2) + s^{13}(\bar{y}_1 - \bar{x}_1)(\bar{y}_3 - \bar{x}_3) \\ & + s^{21}(\bar{y}_2 - \bar{x}_2)(\bar{y}_1 - \bar{x}_1) + s^{22}(\bar{y}_2 - \bar{x}_2)(\bar{y}_2 - \bar{x}_2) + s^{23}(\bar{y}_2 - \bar{x}_2)(\bar{y}_3 - \bar{x}_3) \\ & + s^{31}(\bar{y}_3 - \bar{x}_3)(\bar{y}_1 - \bar{x}_1) + s^{32}(\bar{y}_3 - \bar{x}_3)(\bar{y}_2 - \bar{x}_2) + s^{33}(\bar{y}_3 - \bar{x}_3)(\bar{y}_3 - \bar{x}_3) \\ & = 1.5764. \end{aligned}$$

$$\frac{\bar{\alpha}_2 - \bar{\alpha}_1}{2\bar{\sigma}} = .792$$

$$P_1 = 1 - P_2 = \frac{1}{\sqrt{2\pi}} \int_{.792}^{\infty} e^{-t^2/2} = .2062$$

where P_1 is the probability of making an error of Type I, that is, of classifying an individual as one who will do satisfactory work when he actually does unsatisfactory work; and $1 - P_2$ is the probability of making an error of Type II,

that is, of classifying a student as one who will do unsatisfactory work when he actually does satisfactory work.

3. Conclusions. In using the above classification equation to classify the 305 trainees used in this study, 21 errors of Type I were made or 22.9 percent, while 50 errors of Type II were made or 23.9 percent. These percentages seem reasonably close to the expected 20.6 percent.

NOTE ON AN IDENTITY IN THE INCOMPLETE BETA FUNCTION

By T. A. Bancroft

Iowa State College

Since the incomplete beta function has proved of some importance in statistics, it would appear that any additional information concerning its properties might at some time prove useful. In a paper by the author, [1], two identities in the incomplete beta function were incidentally obtained. They are as follows:

$$(1)\qquad (p+q)I_x(p,q) = pI_x(p+1,q) + qI_x(p,q+1)$$

and

$$(2)\qquad (p+q+1)^{[2]}I_x(p,q) = (p+1)^{[2]}I_x(p+2,q) + 2pqI_x(p+1,q+1) + (p+1)^{[2]}I_x(p,q+2),$$

where the incomplete beta function $I_x(p,q) = \dfrac{B_x(p,q)}{B(p,q)}$, etc., and $(p+1)^{[2]}$, etc. refer to the standard factorial notation.

Written in the above form these two identities suggest a possible general identity to which they belong as special cases. The third special case suggested is:

$$(3)\qquad \begin{aligned}(p+q+2)^{[3]}I_x(p,q) &= (p+2)^{[3]}I_x(p+3,q)\\ &+ 3(p+1)^{[2]}qI_x(p+2,q+1) + 3p(q+1)^{[2]}I_x(p+1,q+2)\\ &+ (q+2)^{[3]}I_x(p,q+3).\end{aligned}$$

The general formula suggested is

$$(4)\qquad (p+q+n-1)^{[n]}I_x(p,q) = \sum_{r=0}^{n}\binom{n}{r}(p+n-r-1)^{[n-r]}(q+r-1)^{[r]}I_x(p+n-r,q+r).$$

To prove the general formula we write (4) as

$$(5)\qquad (p+q+n-1)^{[n]}I_x(p,q) = \sum_{r=0}^{n}\binom{n}{r}(p+n-r-1)^{[n-r]}\cdot(q+r-1)^{[r]}\frac{B_x(p+n-r,q+r)}{B(p+n-r,q+r)}.$$

By expanding and simplifying it is easy to show that

$$\frac{(p+n-r-1)^{[n-r]}(q+r-1)^{[r]}}{\mathrm{B}(p+n-r,\, q+r)} = \frac{(p+q+n-1)^{[n]}}{\mathrm{B}(p,\, q)}. \tag{6}$$

Using (6) the right hand side of (5) reduces to

$$\frac{(p+q+n-1)^{[n]}}{\mathrm{B}(p,\, q)} \sum_{r=0}^{n} \binom{n}{r} \mathrm{B}_x(p+n-r,\, q+r). \tag{7}$$

The summed function in (7) reduces to

$$\int_0^x x^{p-1}(1-x)^{q-1}[x+(1-x)]^n\, dx = \mathrm{B}_x(p,\, q), \tag{8}$$

which proves the identity.

Although the general identity is quite simple to prove, it does not seem to have appeared in the literature

REFERENCE

[1] Bancroft, T A. "On biases in estimation due to the use of preliminary tests of significance," *Annals of Math. Stat* , Vol. 15 (1944), No 2

NEWS AND NOTICES

Readers are invited to submit to the Secretary of the Institute news items of interest

Personal Items

Archie Blake is now employed as a ballistician with the Ballistic Research Laboratory at Aberdeen Proving Ground.

Robert V. Bonnar is now employed as Associate Technologist at the Mare Island Navy Yard.

Professor W. G. Cochran has returned to his regular duties at Iowa State College

Mrs Bianca Cody (Bianca Rivoli) is now Statistician for the James O. Peck Research Company, 12 East 41st Street, New York City.

Associate Professor William Feller of Brown University has been appointed Professor of Mathematics at Cornell University.

Professor John Kenney of the University of Wisconsin is now located at the Milwaukee branch of the University.

Myra Levine is now Assistant Mathematical Statistician with the Statistical Research Group at Columbia University.

Mrs. Harold Michaelis (Ruth E. Jolliffe) is 5th Naval District Statistician at the Naval Operating base in Norfolk, Va.

Emma Spaney is Statistician for the Committee on Measurement of the National League of Nursing Education.

Professor J. A. Shohat of the University of Pennsylvania died October 8, 1944.

Mr. Redford T. Webster of the Western Electric Company died July 31, 1944.

New Members

The following persons have been elected to membership in the Institute:

Boddie, John B., Jr. Chief, Program Section, Budget Division, Washington, D. C. *2628 Tunlaw Road, N.W.*

Bruner, Nancy M A. (Iowa) Statistician, Western Auto Supply Co., Kansas City, Mo. *7511 Main St*

Christopher, Edward E. B.S. (Mass. Inst. Tech) Statistician, Signal Corps. *5704 North 26th St., Arlington, Va.*

Cowden, Dudley J. Ph.D. (Columbia) Prof. of Economics, Univ. of North Carolina. *Box 515, Chapel Hill, North Carolina.*

Cynamon, Manuel M.S. (City Coll., N. Y) Personnel Tech., Personnel Res. Sec., Adj. General's Office, War Dept. *10 Ave. P, Brooklyn 4, N. Y.*

Evensen, Edward J. On military leave from Metropolitan Life Ins. Co. (Actuarial Sec.) *Sv. Co., 1st Sp. Sv Force*

Green, Earl L. Ph.D. (Brown) 1st Lieut., A.C., Chief, Dept. of Statistics. AAF School of Aviation Medicine, Randolph Field, Texas.

Groves, William Brewster B S. (Antioch) Economist, Off. of Price Administration. *520 Decatur St , N.W , Washington, D. C.*

Hornseth, Richard Allen M A (Wisconsin) Res Assistant in Sociology, Univ. of Wisconsin. *207 N. Randall, Madison 5, Wis.*

Kinsler, David M. M A. (Chicago) Chief, Analytical Section, Arms & Ammunition Division, Aberdeen Proving Ground, Maryland

Kopp, Paul J. M A. (Duke) Major, Chemical Warfare Service, U. S. A. *1305 North Adams St., Arlington, Va.*

Massey, Frank Jones, Jr. M.A (California) Associate, Dept. of Math , Univ. of California, Berkeley, Calif. *1364 Union St , San Francisco 9, Calif.*

Orcutt, Guy H. Ph.D. (Michigan) Instr. Economics Dept., Mass. Inst. of Tech., Cambridge, Mass.

Rakesky, Sophie M.S. (Michigan) Statistician, W. K Kellogg Foundation, Battle Creek, Mich.

Roberts, Jean M.S. (Minnesota) Statistician, Child Welfare Res. Analyst. *929 Goodrich Ave., St Paul 5, Minn.*

Schietroma, William B.S.S. (Coll. of City of N Y.) Research Assistant. *316 East 116th St , New York, N Y.*

Schlorek, Mary A. A B. (Adelphi) Research Statistician, National Broadcasting Co., 30 Rockefeller Plaza, New York, N. Y.

deSousa, Alvaro Pedro B.E. (Liverpool) Vice-Governor, Banco de Portugal. *Monserrate, Rua Infante de Sagres, Estoril, Portugal.*

Steele, Floyd George M S (Calif. Inst. of Tech.) Stat. Analyst, Douglas Aircraft. *18168 Roosevelt Highway, Pacific Palisades, Calif*

Thom, Herbert C. S. 6130 18th Rd , N., Arlington, Va.

Report of the Fifth Pittsburgh Chapter Meeting

The fifth meeting of the Pittsburgh Chapter of the Institute of Mathematical Statistics was held at Engineering Hall, Carnegie Institute of Technology on Saturday, November 25, 1944. The meeting was held as a joint session with the Pittsburgh Quality Control Society. Thirty-one persons attended the meeting, including the following six members of the Institute:

George Eldredge, H. J. Hand, C. R. Mummery, E G Olds, E. M. Schrock, J. V. Sturtevant.

The following papers were presented, with Mr. J. V. Sturtevant, of the Carnegie Illinois Steel Corporation, acting as chairman:

1. *Modified Application of Control Chart to the Use of Gauges on Machine Tool Work.*
   Dr E. G. Olds, War Production Board, Washington, D. C.
2. *Application of Control Charts to Infrequent Inspection of Machine Operations.*
   W. D. Angst, Thompson Aircraft Products Company, Cleveland, Ohio.
3. *Application of Control Chart Techniques to Checking Reproducibility of Chemical Analysis.*
   H. A. Stobbs, Wheeling Steel Corporation, Steubenville, Ohio.
4. *Statistical Principles of Experimental Design as Applied to Tests Conducted in Manufacturing Operations.*
   Dr. B Epstein, Westinghouse Electric & Manufacturing Co , East Pittsburgh, Pa.

H. J. HAND,
Secretary-Treasurer, Pittsburgh Chapter

Educational Meetings of the Pittsburgh Chapter

The first of a series of educational meetings on methods of statistical computations given by the Pittsburgh Chapter was held on Saturday afternoon, January 20, 1945. Thirty-three persons attended the meeting, including the following three members of the Institute:

Thomas A. Elkins, H. J. Hand, J. V. Sturtevant.

The following program was presented:

1. *Potential Field for Industrial Applications of Statistical Method.*
   H. J. Hand, National Tube Company, Pittsburgh, Pa.
2. *Computations for Analysis of Variance and Experimental Design.*
   Ben Epstein, Westinghouse Electric & Manufacturing Company, East Pittsburgh, Pa.

It is planned to hold these meetings bi-weekly, on Saturday afternoons for an indefinite period in the future. Topics to be considered in the series will include:

1. Analysis of variance and covariance.
2. Design of experiments.
3. Tests of significance.
4. Probability and probability distributions.
5. Correlation and regression analysis, including the orthogonal coordinate method.
6. Tests of increased severity.
7. Sampling theory, including stratification.
8. Acceptance-rejection mathematics, Dodge sampling inspection tables.
9. Shewhart control chart techniques.
10. Analysis of runs.
11. Cycle analysis.
12. Factor analysis.

H. J. HAND,
Secretary-Treasurer, Pittsburgh Chapter

ANNUAL REPORT OF THE PRESIDENT OF THE INSTITUTE

Continuing the established tradition, the annual summer meeting was held at Wellesley, Massachusetts, August 12–13, 1944 in conjunction with the Summer Meetings of the American Mathematical Society and the Mathematical Association of America. A regional meeting was held in Washington, May 6–7, in conjunction with the meeting of the Washington Chapter of the American Statistical Association. The programs were arranged by the Program Committee: W. Feller, Chairman, W. G. Madow, and A. Wald.

Even though, under present war conditions, research in the field of probability and statistics is very much curtailed, enough papers in mathematical statistics of satisfactory quality have been proposed for publication in the *Annals* in 1944 to keep the total volume of material at approximately five hundred pages or the level of the last few years. However, the outlook for a sufficient number of satisfactory papers to maintain the usual volume of publication during 1945 does not look quite so favorable.

Looking into the future, the Institute must continue to furnish, through the *Annals*, a medium for the publication of all important results of original research in the field of mathematical statistics as they become available. To do otherwise would be suicide. At the same time we must take account of the growing need for comprehensive surveys of statistical theory on the part of other scientists, including not only social scientists but also physicists, chemists, biologists, and research engineers, whose interest in the contributions of mathematical statistics has been greatly stimulated during the war. Only the mathematical statiscian of broad competence can provide adequate critical surveys of this character. Perhaps some of this need can be met through survey articles published in the *Annals*, although it is not an easy matter to get capable men to do such work. Perhaps the time is not far off when the Institute must stimulate the preparation of such material by instituting an annual series of Colloquium Lectures patterned somewhat after those of the Mathematical Society, which could be published separately.

This is but one of many problems that the Institute faces in its post-war development. Not only must it assume the responsibility of stimulating and encouraging research and of publishing the results; it must also consider the problem of training the research statistician of tomorrow as well as those who are to apply mathematical statistics in the many fields of science. It also must assume some responsibility for keeping in contact with other scientists in order that the mathematical statistician may become acquainted with the unsolved statistical problems of the scientist. There are also many problems of a professional character that face the mathematical statistician in the future if he is to succeed in developing the profession of mathematical statistics to the level attained by some of the older scientific professions.

With the realization of the need for a concerted attack on some of these

problems, the Board of Directors at its meeting in May set up two committees, one on Training and Placement of Statisticians under Harold Hotelling and the other on Post-War Development of the Institute under W. G. Cochran. Interim reports received by the Board from both committees indicate that considerable progress has been made to date. They also indicate, however, that much more work remains to be done

At the same meeting of the Board, a Budget and Finance Committee was set up, consisting of P. S. Dwyer, Chairman, C. H. Fischer, A. C. Olshen, and C. F. Roos, to prepare a report on the policy that should be followed by the Institute in respect to such items as investment of funds, advertising, preparation of an annual statement, and the like. Some of the work of this committee has already borne fruit, as, for example, in providing the actuarial basis for life membership adopted at the Wellesley meeting and in establishing certain principles to be used in conducting the business of the Institute

A report of the Committee on Membership, W. G. Cochran, Chairman, P. S. Dwyer, and T. Koopmans, appears elsewhere in this issue of the *Annals*. Upon recommendation of this committee, the Board of Directors elected nine new fellows: Walter Bartky, C. I. Bliss, Gertrude M. Cox, P. A. Horst, M. G. Kendall, H. B. Mann, E. S. Pearson, Henry Scheffé, and W. A. Wallis.

The nominating committee for the year consisted of John Curtiss, Chairman, E. G. Olds, and F. F. Stephan. G. W. Snedecor served the Institute again as its representative on the Council of the A.A.A.S.

The annual election of the Institute just concluded by mail ballot resulted in the election of the following officers for 1945: W. E. Deming, President; W. G. Cochran, and J. L. Doob, Vice-Presidents.

WALTER A. SHEWHART
President, 1944

February 10, 1945

ANNUAL REPORT OF THE SECRETARY-TREASURER OF THE INSTITUTE

Accounts of the 1944 meetings of the Institute—the Wellesley meeting, the Washington regional meeting, and the Pittsburgh chapter meetings—have appeared in appropriate issues of the *Annals*

At the Wellesley meeting a number of amendments to the Constitution and By-Laws were passed These were published in the September, 1944, issue of the *Annals*. (The amended Constitution and By-Laws appear elsewhere in this issue.)

Due to a large extent to the cooperation of the membership in sending in nominations, the Institute enjoyed a large increase in membership during the year. There were some resignations and it was necessary to suspend fifteen persons at the end of 1944 because of failure to pay dues. It is apparent that, in some of these cases at least, our mail is not being received. Undoubtedly some of these memberships will be restored when contact is again established. As of January 1, 1945, there were 606 members, a net gain of approximately one hundred members.

During the year the Institute received gifts from Professor Harry Carver in the form of exchanges for early issues of the *Annals*, reprints of early articles, etc.

The Secretary-Treasurer wishes to acknowledge the continued assistance of Professor Lloyd Knowler in looking after the back issues of the *Annals* which are stored at Iowa City.

The following financial statement covers the period from December 22, 1943 to December 31, 1944 (the books and records of the Treasurer have been audited by Professor Thomas A Bickerstaff and were found to be in agreement with the statement as submitted):

FINANCIAL STATEMENT

December 22, 1943, to December 31, 1944

RECEIPTS

BALANCE ON HAND, DECEMBER 22, 1943		$3,715.05
DUES		
1944 and before	$2,905.31	
1945 and 1946	1,127.00	
Life	330.00	
		4,452.31
SUBSCRIPTIONS		
1944 and before	$1,301.94	
1945 and 1946	883.94	
		2,185.88
SALE OF BACK NUMBERS		1,385.02
MISCELLANEOUS		6.15
Total Receipts		$11,744.41

EXPENDITURE

ANNALS—CURRENT		
Office of Editor	$273.77	
Waverly Press	3,448.51	
		3,722.28
ANNALS—BACK NUMBERS		
Purchase from H. C. Carver	$149.40	
Iowa City Office	96.26	
		245.66
OFFICE OF SECRETARY-TREASURER		
Printing, mimeographing, programs, etc. (including stamped envelopes)	$377.00	
Postage and supplies	68.02	
Clerical help	455.94	
Moving office from Pittsburgh	55.79	
		956.75
MISCELLANEOUS		29.07
BALANCE ON HAND, DECEMBER 31, 1944		6,790 65
		$11,744.41

No unpaid bills were in the hands of the Treasurer as of December 31, 1944, and aside from an additional $100.00 which the Board has designated for *Annals* expense for 1944, there were no large bills outstanding.

Accounts receivable as of December 31, 1944, amounted to $303.73. Many of these accounts are current accounts while some of the older ones are accounts with firms in India, which probably will be collected eventually.

The American Library Association continued with its purchase of thirty sets of Volume XV of the *Annals* (for post war distribution) and the Universal Trading Corporation (representing the Chinese Government) purchased twenty sets of Volumes 11–17 inclusive. These orders contributed in no small way to the total 1944 income of $8,029.36.

The 1944 balance $6,790.65 (consisting of bank balance of $3,790.65 and $3,000 00 in government bonds) is $3,075.60 higher than it was on December 21, 1943. This increase is due in part to 1944 business and in part to the fact that unusually large payments toward future business, such as the $330.00 in life payments and the $1,127.00 in 1945 and 1946 dues, have been made.

To summarize the situation briefly, the Institute's 1944 activity has resulted in a gain of approximately $1,500.00 and we are about this much in advance of our usual position with reference to the payments of following years.

PAUL S. DWYER
Secretary-Treasurer.

December 31, 1944

[OF THE MEMBERSHIP COMMITTEE OF THE INSTITUTE

e duties of this Committee are not defined in detail in the Constitution, of Directors asked the Committee to prepare a statement describing priate composition and function of the Committee on Membership. resulted in the preparation of amendments to the Constitution and These amendments were passed at the business meeting at Wellesley August 13, 1944, and are printed in full in the September, 1944, issue *ıals* (p. 340).

, the duties of the Committee are specified as follows in these amend-

: Committee holds the power of election to the grades of Member and ember and makes recommendations to the Board of Directors with to placing members in the other grades of membership.

s the duty of the Committee to prepare and make available through tary-Treasurer an announcement of the qualifications necessary for nt grades of membership and to review these qualifications periodically.

) Committee considers plans for increasing the number of applicants ership.

ıitted by the amendments referred to above, the power of election to s of Member and Junior Member was delegated by the Committee in 944, to the Secretary-Treasurer, subject to certain reservations. The of qualifications for the different grades of membership as mentioned ›ve is published below. At the August 13 meeting of the Board of it was decided that no elections should be made at present to the grades ry Member and Sustaining Member.

recommendation of the Membership Committee the following members ed as Fellows by the Board of Directors: W. Bartky, C. I. Bliss, G. M. . Horst, M. G. Kendall, H. B. Mann, E. S. Pearson, H. Scheffé, W. A.

Statement of Qualifications for the Different Grades of Membership in the Institute of Mathematical Statistics

'. The candidate shall either (a) be actively engaged in or show a :erest in mathematical statistics, or (b) be interested in some applied ·tistics, with a desire to keep himself informed regarding recent develop- nathematical theory and techniques.

Member.

undergraduate student of a collegiate institution is eligible for election r Member of the Institute of Mathematical Statistics provided that he ponsored by a member of the Institute.

annual dues ($2.50) must be submitted with the application.

3. Annual membership shall coincide with the calendar year and the Junior Member shall receive a complete volume of the *Annals of Mathematical Statistics* for the year in which he or she is elected.

4 Junior Membership shall be limited to a term of two years, but a Junior Member may apply for transfer to ordinary membership at the beginning of his second year.

Fellow.

1. The candidate shall have evidenced continuing activity in research in mathematical statistics by publication beyond his doctor's dissertation of independent work of merit. Normally two or three worthwhile papers beyond the dissertation will be required to establish this fact.

2. The first qualification may be partly or wholly waived in the case of (a) a candidate of well-established leadership among mathematical statisticians whose contributions to the development of the field of mathematical statistics other than sufficient published original research shall be judged of equal value or (b) a candidate of well-established leadership in the applications of mathematical statistics, whose work has contributed greatly to the utility of and the appreciation for mathematical statistics.

Honorary Member. A person of exceptional ability and acknowledged leadership in the field of mathematical statistics may be elected to the grade of Honorary Member by the Board of Directors, upon the recommendation of the Committee on Membership.

Sustaining Member. The Board of Directors shall have the power to elect to Sustaining Membership any individual, group or corporation that is interested in furthering the purposes for which the Institute was formed.

W. G. COCHRAN (*Chairman*)
W. E. DEMING
P. S. DWYER
T. KOOPMANS

February 10, 1945

PROGRESS REPORT OF THE COMMITTEE ON POST-WAR DEVELOPMENT OF THE INSTITUTE

In considering the post-war development of the Institute of Mathematical Statistics, the Committee has recognized two general problems:

A. The problem of what additional activities the Institute should undertake in order to provide further stimulus to the development of the field of mathematical statistics.

B. The problem of determining how the Institute can cooperate more effectively with the users of statistical techniques.

Because of rapidly increasing interest in the application of statistical methods in many different fields, the Committee has directed most of its attention thus far to Problem B; the present progress report is concerned with the work of the Committee on this problem. The Committee hopes to submit a report on Problem A at the end of 1945.

With respect to Problem B, it is the opinion of the Committee that a central organization for the statistical societies should be of common interest. Accordingly, a plan was worked out and submitted to the Board of Directors of the Institute at the Wellesley meeting of the Institute. This proposal and its present status are discussed below.

We believe that there is much to be gained from an organization that would form a link between the various statistical societies, and would have the following principal aims:

(1) To represent the members of the societies in all matters of common interest.

(2) To promote cooperation between statisticians working in the different fields of application, and between mathematical statistics, applied statistics, scientific research and the industries.

(3) To develop amongst the public an appreciation of the value of the statistical method in scientific inquiry.

It is our opinion that an organization similar to that of the Institute of Physics would be suitable The statistical societies, while retaining their present autonomies, would become founding members of a corporation whose governing board would contain representatives from each society. In pursuance of its aims as outlined above, the new organization might:

(a) Take the lead in formulating policies on questions which concern all statisticians.

(b) Publish a journal of general interest to statisticians and undertake the routine work in connection with the publication of the journals of the individual societies, the societies retaining in full their present responsibility for the contents of their journals.

(c) Arrange joint meetings between different statistical societies and between statistical and other scientific societies.

(d) Assist new groups in organizing for their benefit, either under the auspices of one of the present societies or in a new society, which might at first be given associate membership and later full membership of the central organization.

(e) Take steps to bring news about the use of statistics in scientific research to the attention of the public and more particularly of leaders in industry, in federal, state and local agencies and in education.

(f) Investigate the demands for various types and degrees of statistical training, outline courses of training in statistics suitable for meeting these demands and make strenuous efforts to have the recommended courses of training put into effect, in order that statisticians can be of fullest service in the nation's work. In this connection an information and placement bureau may be an appropriate auxiliary.

(g) Institute an abstracting service in statistical methodology. This might take the form of a periodical publication of abstracts of papers with respect to their methodological content rather than their subject matter. The coverage would include journals of business, marketing, engineering, medicine and agriculture as well as purely statistical publications.

The financial needs of the new organization, which would maintain a paid full-time staff, may be met initially by contributions from the present societies. In view of the extra services which would be rendered to statisticians, some increase in the subscription rates of the present societies appears reasonable. A member who belongs to more than one of the present societies would pay the extra amount only once. Supplementary income might be derived from advertising in the journal of the central organization and from the establishment of sustaining or corporate memberships in the central organization.

At the time of the Wellesley meeting of the Board, there had been only informal contacts between members of this Committee and members of other statistical societies. We considered it our first task to obtain some consensus of opinion from the standpoint of the Institute of Mathematical Statistics. Following general approval by the Board of Directors of the Institute, members of the Committee discussed the proposal for a central organization with representatives of several other statistical societies. The American Statistical Association has a Committee to consider the future structure of the Association and this Committee brought the Institute proposal before the Board of Directors of the Association for action. As the oldest of the statistical societies, the American Statistical Association then invited participation in an intersociety committee by the Institute and nine other societies or sections, directly or indirectly concerned with statistical method. This committee is to explore the possibilities of coordinating the activities of the several statistical societies and report its recommendations back to each organization. The representatives have now been named and the first meeting was held on February 10, 1945, in New York. At this meeting the Institute was represented by W. G. Cochran and Lt. John H. Curtiss.

With regard to the problem of what additional activities the Institute should undertake in order to furnish additional stimulation to the development of the field of mathematical statistics, the Committee has discussed several ideas which appear promising. It is hoped to present a complete report on this phase of the Committee's work at the end of this year

C. I. Bliss
W. G. Cochran (*Chairman*)
W. E. Deming
P. S. Olmstead
S. S. Wilks

February 12, 1945

CONSTITUTION

OF THE

INSTITUTE OF MATHEMATICAL STATISTICS

ARTICLE I

Name and Purpose

1. This organization shall be known as the Institute of Mathematical Statistics.
2. Its object shall be to promote the interests of mathematical statistics.

ARTICLE II

Membership

1. The membership of the Institute shall consist of Members, Junior Members, Fellows, Honorary Members, and Sustaining Members.
2. Voting members of the Institute shall be (a) the Fellows, and (b) all others, Junior Members excepted, who have been members for twenty-three months prior to the date of voting.
3. No person shall be a Junior Member of the Institute for more than a limited term as determined by the Committee on Membership and approved by the Board of Directors.

ARTICLE III

Officers, Board of Directors, and Committee on Membership

1. The Officers of the Institute shall be a President, two Vice-Presidents, and a Secretary-Treasurer. The terms of office of the President and Vice-Presidents shall be one year and that of the Secretary-Treasurer three years. Elections shall be by majority ballots at Annual Meetings of the Institute. Voting may be in person or by mail.

(a) Exception. The first group of Officers shall be elected by a majority vote of the individuals present at the organization meeting, and shall serve until December 31, 1936.

2. The Board of Directors of the Institute shall consist of the Officers, the two previous Presidents, and the Editor of the Official Journal of the Institute.

3 The Institute shall have a Committee on Membership composed of a Chairman and three Fellows. At their first meeting subsequent to the Adoption of this Constitution, the Board of Directors shall elect three members as Fellows to serve as the Committee on Membership, one member of the Committee for a term of one year, another for a term of two years, and another for a term of three years. Thereafter the Board of Directors shall elect from among the Fellows one member annually at their first meeting after their election for a term of three years. The president shall designate one of the Vice-Presidents as Chairman of this Committee.

ARTICLE IV

Meetings

1. A meeting for the presentation and discussion of papers, for the election of Officers, and for the transaction of other business of the Institute shall be held annually at such time as the Board of Directors may designate. Additional meetings may be called from

time to time by the Board of Directors and shall be called at any time by the President upon written request from ten Fellows. Notice of the time and place of meeting shall be given to the membership by the Secretary-Treasurer at least thirty days prior to the date set for the meeting. All meetings except executive sessions shall be open to the public Only papers accepted by a Program Committee appointed by the President may be presented to the Institute.

2 The Board of Directors shall hold a meeting immediately after their election and again immediately before the expiration of their term. Other meetings of the Board may be held from time to time at the call of the President or any two members of the Board. Notice of each meeting of the Board, other than the two regular meetings, together with a statement of the business to be brought before the meeting, must be given to the members of the Board by the Secretary-Treasurer at least five days prior to the date set therefor. Should other business be passed upon, any member of the Board shall have the right to reopen the question at the next meeting.

3. Meetings of the Committee on Membership may be held from time to time at the call of the Chairman or any member of the Committee provided notice of such call and the purpose of the meeting is given to the members of the Committee by the Secretary-Treasurer at least five days before the date set therefor. Should other business be passed upon, any member of the Committee shall have the right to reopen the question at the next meeting. Committee business may also be transacted by correspondence if that seems preferable.

4. At a regularly convened meeting of the Board of Directors, four members shall constitute a quorum. At a regularly convened meeting of the Committee on Membership, two members shall constitute a quorum.

ARTICLE V

Publications

1. The *Annals of Mathematical Statistics* shall be the Official Journal for the Institute. The Editor of the *Annals of Mathematical Statistics* shall be a Fellow appointed by the Board of Directors of the Institute. The term of office of the Editor may be terminated at the discretion of the Board of Directors.

2. Other publications may be originated by the Board of Directors as occasion arises.

ARTICLE VI

Expulsion or Suspension

1. Except for non-payment of dues, no one shall be expelled or suspended except by action of the Board of Directors with not more than one negative vote.

ARTICLE VII

Amendments

1. This constitution may be amended by an affirmative two-thirds vote at any regularly convened meeting of the Institute provided notice of such proposed amendment shall have been sent to each voting member by the Secretary-Treasurer at least thirty days before the date of the meeting at which the proposal is to be acted upon. Voting may be in person or by mail.

BY-LAWS

ARTICLE I

Duties of the Officers, the Editor, Board of Directors, and Committee on Membership

1. The President, or in his absence, one of the Vice-Presidents, or in the absence of the President and both Vice-Presidents, a Fellow selected by vote of the Fellows present, shall preside at the meetings of the Institute and of the Board of Directors. At meetings of the Institute, the presiding officer shall vote only in the case of a tie, but at meetings of the Board of Directors he may vote in all cases. At least three months before the date of the annual meeting, the President shall appoint a Nominating Committee of three members. It shall be the duty of the Nominating Committee to make nominations for Officers to be elected at the annual meeting and the Secretary-Treasurer shall notify all voting members at least thirty days before the annual meeting. Additional nominations may be submitted in writing, if signed by at least ten Fellows of the Institute, up to the time of the meeting.

2. The Secretary-Treasurer shall keep a full and accurate record of the proceedings at the meetings of the Institute and of the Board of Directors, send out calls for said meetings and, with the approval of the President and the Board, carry on the correspondence of the Institute. Subject to the direction of the Board, he shall have charge of the archives and other tangible and intangible property of the Institute, and once a year he shall publish in the *Annals of Mathematical Statistics* a classified list of all Members and Fellows of the Institute. He shall send out calls for annual dues and acknowledge receipt of same; pay all bills approved by the President for expenditures authorized by the Board or the Institute; keep a detailed account of all receipts and expenditures, prepare a financial statement at the end of each year and present an abstract of the same at the annual meeting of the Institute after it has been audited by a Member or Fellow of the Institute appointed by the President as Auditor. The Auditor shall report to the President.

3. Subject to the direction of the Board, the Editor shall be charged with the responsibility for all editorial matters concerning the editing of the *Annals of Mathematical Statistics*. He shall, with the advice and consent of the Board, appoint an Editorial Committee of not less than twelve members to co-operate with him; four for a period of five years, four for a period of three years, and the remaining members for a period of two years, appointments to be made annually as needed. All appointments to the Editorial Committee shall terminate with the appointment of a new Editor. The Editor shall serve as editorial adviser in the publication of all scientific monographs and pamphlets authorized by the Board.

4. The Board of Directors shall have charge of the funds and of the affairs of the Institute, with the exception of those affairs specifically assigned to the President or to the Committee on Membership. The Board shall have authority to fill all vacancies ad interim, occurring among the Officers, Board of Directors, or in any of the Committees. The Board may appoint such other committees as may be required from time to time to carry on the affairs of the Institute. The power of election to the different grades of Membership, except the grades of Member and Junior Member, shall reside in the Board.

5. The Committee on Membership shall prepare and make available through the Secretary-Treasurer an announcement indicating the qualifications requisite for the differ-

ent grades of membership. The Committee shall review these qualifications periodically and shall make such changes in these qualifications and make such recommendations with reference to the number of grades of membership as it deems advisable. The power to elect worthy applicants to the grades of Member and Junior Member shall reside in the Committee, which may delegate this power to the Secretary-Treasurer, subject to such reservations as the Committee considers appropriate. The Committee shall make recommendations to the Board of Directors with reference to placing members in other grades of membership. The Committee shall give its attention to the question of increasing the number of applicants for membership and shall advise the Secretary-Treasurer on plans for that purpose.

ARTICLE II

Dues

1. Members shall pay five dollars at the time of admission to membership and shall receive the full current volume of the Official Journal. Thereafter, Members shall pay five dollars annual dues. The annual dues of Junior Members shall be two dollars and fifty cents. The annual dues of Fellows shall be five dollars. The annual dues of Sustaining Members shall be fifty dollars. Honorary Members shall be exempt from all dues.

(a) Exception In the case that two Members of the Institute are husband and wife and they elect to receive between them only one copy of the Official Journal, the annual dues of each shall be three dollars and seventy-five cents.

(b) Exception. Any Member or Fellow may make a single payment which will be accepted by the Institute in place of all succeeding yearly dues and which will not otherwise alter his status as a Member or Fellow. The amount of this payment will depend upon the age of this Member or Fellow and will be based upon a suitable table and rate of interest, to be specified by the Board of Directors.

(c) Exception. Any Member or Junior Member of the Institute serving, except as a commissioned officer, in the Armed Forces of the United States or of one of its allies, may upon notification to the Secretary-Treasurer be excused from the payment of dues until the January first following his discharge from the Service. He shall have all privileges of membership except that he shall not receive the Official Journal. However during the first year of his resumed regular membership he may have the right to purchase, at $2.50 per volume, one copy of each volume of the Official Journal published during the period of his service membership.

2. Annual dues shall be payable on the first day of January of each year.

3. The annual dues of a Fellow, Member, or Junior Member include a subscription to the Official Journal. The annual dues of a Sustaining Member include two subscriptions to the Official Journal.

4. It shall be the duty of the Secretary-Treasurer to notify by mail anyone whose dues may be six months in arrears, and to accompany such notice by a copy of this Article. If such person fail to pay such dues within three months from the date of mailing such notice, the Secretary-Treasurer shall report the delinquent one to the Board of Directors, by whom the person's name may be stricken from the rolls and all privileges of membership withdrawn. Such person may, however, be re-instated by the Board of Directors upon payment of the arrears of dues.

ARTICLE III

Salaries

1. The Institute shall not pay a salary to any Officer, Director, or member of any committee.

ARTICLE IV

Amendments

1. These By-Laws may be amended in the same manner as the Constitution or by a majority vote at any regularly convened meeting of the Institute, if the proposed amendment has been previously approved by the Board of Directors.

SEQUENTIAL TESTS OF STATISTICAL HYPOTHESES

By A. Wald

Columbia University

Table of Contents

A. Introduction

By a sequential test of a statistical hypothesis is meant any statistical test procedure which gives a specific rule, at any stage of the experiment (at the n-th trial for each integral value of n), for making one of the following three decisions: (1) to accept the hypothesis being tested (null hypothesis), (2) to reject the null hypothesis, (3) to continue the experiment by making an additional observation. Thus, such a test procedure is carried out sequentially. On the basis of the first trial, one of the three decisions mentioned above is made. If the first or the second decision is made, the process is terminated. If the third decision is made, a second trial is performed. Again on the basis of the first two trials one of the three decisions is made and if the third decision is reached a third trial is performed, etc. This process is continued until either the first or the second decision is made.

An essential feature of the sequential test, as distinguished from the current test procedure, is that the number of observations required by the sequential test is not predetermined, but is a random variable due to the fact that at any stage of the experiment the decision of terminating the process depends on the results of the observations previously made. The current test procedure may be considered a limiting case of a sequential test in the following sense: For any positive integer n less than some fixed positive integer N, the third decision is always taken at the n-th trial irrespective of the results of these first n trials. At the N-th trial either the first or the second decision is taken. Which decision is taken will depend, of course, on the results of the N trials.

In a sequential test, as well as in the current test procedure, we may commit two kinds of errors. We may reject the null hypothesis when it is true (error of the first kind), or we may accept the null hypothesis when some alternative hypothesis is true (error of the second kind). Suppose that we wish to test the null hypothesis H_0 against a single alternative hypothesis H_1, and that we want the test procedure to be such that the probability of making an error of the first kind (rejecting H_0 when H_0 is true) does not exceed a preassigned value α, and the probability of making an error of the second kind (accepting H_0 when H_1 is true) does not exceed a preassigned value β. Using the current test procedure, i.e., a most powerful test for testing H_0 against H_1 in the sense of the Neyman-Pearson theory, the minimum number of observations required by the test can be determined as follows: For any given number N of observations a most powerful test is considered for which the probability of an error of the first kind is equal to α. Let $\beta(N)$ denote the probability of an error of the second kind for this test procedure. Then the minimum number of observations is equal to the smallest positive integer N for which $\beta(N) \leq \beta$.

In this paper a particular test procedure, called the sequential probability ratio test, is devised and shown to have certain optimum properties (see section 4.7). The sequential probability ratio test in general requires an expected number of observations considerably smaller than the fixed number of observations needed by the current most powerful test which controls the errors of the first

and second kinds to exactly the same extent (has the same α and β) as the sequential test. The sequential probability ratio test frequently results in a saving of about 50% in the number of observations as compared with the current most powerful test. Another surprising feature of the sequential probability ratio test is that the test can be carried out without determining any probability distributions whatsoever. In the current procedure the test can be carried out only if the probability distribution of the statistic on which the test is based is known. This is not necessary in the application of the sequential probability ratio test, and only simple algebraic operations are needed for carrying it out. Distribution problems arise in connection with the sequential probability ratio test only if we want to make statements about the probability distribution of the number of observations required by the test.

This paper consists of two parts. Part I deals with the theory of sequential tests for testing a simple hypothesis against a single alternative. In Part II a theory of sequential tests for testing simple or composite hypotheses against infinite sets of alternatives is outlined. The extension of the probability ratio test to the case of testing a simple hypothesis against a set of one-sided alternatives is straight forward and does not present any difficulty. Applications to testing the means of binomial and normal distributions, as well as to testing double dichotomies are given. The theory of sequential tests of hypotheses with no restrictions on the possible values of the unknown parameters is, however, not as simple. There are several unsolved problems in this case and it is hoped that the general ideas outlined in Part II will stimulate further research.

Sections 5.2, 5.3 and 5.4 in Part II deal with the applications of the sequential probability ratio test to binomial distributions, double dichotomies and normal distributions. These sections are nearly self-contained and can be understood without reading the rest of the paper Thus, readers who are primarily interested in these special cases of the sequential probability ratio test rather than in the general theory, may profitably read only the above mentioned sections For the benefit of readers who lack a sufficient background in the mathematical theory of statistics the exposition in sections 5.2, 5.3 and 5 4 is kept on a fairly elementary level.

It should be pointed sut that whenever the number of observations on which the test is based is for some reason determined in advance, for instance, if certain data are available from past history and no additional data can be obtained, then the current most powerful test procedure is preferable. The superiority of the sequential probability ratio test is due to the fact that it requires a smaller expected number of observations than the current most powerful test. This feature of the sequential probability ratio test is, however, of no value if the number of observations is for some reason determined in advance.

B. Historical Note

To the best of the author's knowledge the first idea of a sequential test, i.e., a test where the number of observations is not predetermined but is dependent

on the outcome of the observations, goes back to H. F. Dodge and H. G. Romig who proposed a double sampling inspection procedure [1]. In this double sampling scheme the decision whether a second sample should be drawn or not depends on the outcome of the observations in the first sample. The reason for introducing a double sampling method was, of course, the recognition of the fact that double sampling results in a reduction of the amount of inspection as compared with "single" sampling.

The double sampling method does not fully take advantage of sequential analysis, since it does not allow for more than two samples. A multiple sampling scheme for the particular case of testing the mean of a binomial distribution was proposed and discussed by Walter Bartky [2]. His procedure is closely related to the test which results from the application of the sequential probability ratio test to testing the mean of a binomial distribution. Bartky clearly recognized the fact that multiple sampling results in a considerable reduction of the average amount of inspection.

The idea of chain experiments discussed briefly by Harold Hotelling [3] is also somewhat related to our notion of sequential analysis. An interesting example of such a chain of experiments is the series of sample censuses of area of jute in Bengal carried out under the direction of P. C. Mahalanobis [6]. The successive preliminary censuses, steadily increasing in size, were primarily designed to obtain some information as to the parameters to be estimated so that an efficient design could be set up for the final sampling of the whole immense jute area in the province.

In March 1943, the problem of sequential analysis arose in the Statistical Research Group, Columbia University,[1] in connection with a specific question posed by Captain G. L. Schuyler of the Bureau of Ordnance, Navy Department. It was pointed out by Milton Friedman and W. Allen Wallis that the mere notion of sequential analysis could slightly improve the efficiency of some current most powerful tests. This can be seen as follows: Suppose that N is the planned number of trials and W_N is a most powerful critical region based on N observations. If it happens that on the basis of the first n trials ($n < N$) it is already certain that the completed set of N trials must lead to a rejection of the null hypothesis, we can terminate the experiment at the n-th trial and thus save some observations. For instance, if W_N is defined by the inequality $x_1^2 + \dots + x_N^2 \geq c$, and if for some $n < N$ we find that $x_1^2 + \dots + x_n^2 \geq c$, we can terminate the process at this stage. Realization of this naturally led Friedman and Wallis to the conjecture that modifications of current tests may exist which take advantage of sequential procedure and effect substantial improvements. More specifically, Friedman and Wallis conjectured that a sequential test may exist that controls the errors of the first and second kinds to exactly the same extent as the current

[1] The Statistical Research Group operates under a contract with the Office of Scientific Research and Development and is directed by the Applied Mathematics Panel of the National Defense Research Committee

most powerful test, and at the same time requires an expected number of observations substantially smaller than the number of observations required by the current most powerful test.[2]

It was at this stage that the problem was called to the attention of the author of the present paper. Since infinitely many sequential test procedures exist, the first and basic problem was, of course, to find the particular sequential test procedure which is most efficient, i.e , which effects the greatest possible saving in the expected number of observations as compared with any other (sequential or non-sequential) test. In April, 1943 the author devised such a test, called the sequential probability ratio test, which for all practical purposes is most efficient when used for testing a simple hypothesis H_0 against a single alternative H_1.

Because of the substantial savings in the expected number of observations effected by the sequential probability ratio test, and because of the simplicity of this test procedure in practical applications, the National Defense Research Committee considered these developments sufficiently useful for the war effort to make it desirable to keep the results out of the reach of the enemy, at least for a certain period of time. The author was, therefore, requested to submit his findings in a restricted report [7] which was dated September, 1943.[3] In this report the sequential probability ratio test is devised and its mathematical theory is developed. In July 1944 a second report [8] was issued by the Statistical Research Group which gives an elementary non-mathematical exposition of the applications of the sequential probability ratio test, together with charts, tables and computational simplifications to facilitate applications.

Independently of the developments here, G. A. Barnard [9] recognized the merits of a sequential method of testing, i.e., the possibility of a saving in the number of observations as compared with the current most powerful test. He also devised an interesting sequential test for testing double dichotomies, which differs from the one obtained by applying the sequential probability ratio test.

Some further developments in the theory of the sequential probability ratio test took place in 1944. Extending the methods used in [7], C. M. Stockman [10] found the operating characteristic curve of the sequential probability ratio test applied to a binomial distribution. Independently of Stockman, Milton Friedman and George W. Brown (independently of each other) obtained the same result which can be extended to the normal distribution and a few other specific distributions, but is not applicable to more general distributions. The general operating characteristic curve for any sequential probability ratio test was derived by the author [11]. A few months later the author developed a general theory of cumulative sums [4] which gives not only the operating char-

[2] Bartky's multiple sampling scheme [2] for testing the mean of a binomial distribution provides, of course, an example of such a sequential test (see, for example, the remarks on p 377 in [2]). Bartky's results were not known to us at that time, since they were published nearly a year later.

[3] The material was recently released making the present publication possible

acteristic curve for any sequential probability ratio test but also the characteristic function of the number of observations required by the test.

The theory of the sequential probability ratio test as given in the present paper differs considerably from the exposition given in [7], since the new developments in [4] have been taken into account. However, some tables and a few sections of the original report [7] are included in the present paper without any substantial changes.

PART I. SEQUENTIAL TEST OF A SIMPLE HYPOTHESIS AGAINST A SINGLE ALTERNATIVE

1. The Current Test Procedure

Let X be a random variable. In what follows in this and the subsequent sections it will be assumed that the random variable X has either a continuous probability density function or a discrete distribution. Accordingly, by the probability distribution $f(x)$ of a random variable X we shall mean either the probability density function of X or the probability that $X = x$, depending upon whether X is a continuous or a discrete variable. Let the hypothesis H_0 to be tested (null hypothesis) be the statement that the distribution of X is $f_0(x)$. Suppose that H_0 is to be tested against the single alternative hypothesis H_1 that the distribution of X is given by $f_1(x)$.

According to the Neyman-Pearson theory of testing hypotheses a most powerful critical region W_N for testing H_0 against H_1 on the basis of N independent observations $x_1, \cdots, x_N$ on X is given by the set of all sample points $(x_1, \cdots, x_N)$ for which the inequality

$$\frac{f_1(x_1)f_1(x_2)\cdots f_1(x_N)}{f_0(x_1)f_0(x_2)\cdots f_0(x_N)} \geq k \tag{1.1}$$

is fulfilled. The quantity k on the right hand side of (1.1) is a constant and is chosen so that the size of the critical region, i.e., the probability of an error of the first kind should have the required value α.

For a fixed sample size N the probability β of an error of the second kind is a single valued function of α, say $\beta_N(\alpha)$, if a most powerful critical region is used. Thus, if in addition to fixing the value of α it is required that the probability of an error of the second kind should have a preassigned value β, or at least it should not exceed a preassigned value β, we are no longer free to choose the sample size N. The minimum number of observations required by the test satisfying these conditions is equal to the smallest integral value of N for which $\beta_N(\alpha) \leq \beta$.

Thus, the current most powerful test procedure for testing H_0 against H_1 can be briefly stated as follows: We choose as critical region the region defined by (1.1) where the constant k is determined so that the probability of an error of the first kind should have a preassigned value α and N is equal to the smallest integer for which the probability of an error of the second kind does not exceed a preassigned value β.

2. The Sequential Test Procedure: General Definitions

2.1. *Notion of a sequential test.* In current tests of hypotheses the number of observations is treated as a constant for any particular problem. In sequential tests the number of observations is no longer a constant, but a random variable In what follows the symbol n is used for the number of observations required by a sequential test and the symbol N is used when the number of observations is treated as a constant.

Sequential tests can be described as follows· For each positive integer m the m-dimensional sample space M_m is subdivided into three mutually exclusive parts R_m^0, R_m^1 and R_m. After the first observation x_1 has been drawn H_0 is accepted if x_1 lies in R_1^0, H_0 is rejected (i e , H_1 is accepted) if x_1 lies in R_1^1, or a second observation is drawn if x_1 lies in R_1. If the third decision is reached and a second observation x_2 drawn, H_0 is accepted, H_1 is accepted, or a third observation is drawn according as the point (x_1, x_2) lies in R_2^0, R_2^1 or in R_2. If (x_1, x_2) lies in R_2, a third observation x_3 is drawn and one of the three decisions is made according as (x_1, x_2, x_3) lies in R_3^0, R_3^1 or in R_3, etc. This process is stopped when, and only when, either the first decision or the second decision is reached. Let n be the number of observations at which the process is terminated. Then n is a random variable, since the value of n depends on the outcome of the observations. (It will be seen later that the probability is one that the sequential process will be terminated at some finite stage.)

We shall denote by $E_0(n)$ the expected value of n if H_0 is true and by $E_1(n)$ the expected value of n if H_1 is true. These expected values, of course, depend on the sequential test used. In order to put this dependence in evidence, we shall occasionally use the symbols $E_0(n \mid S)$ and $E_1(n \mid S)$ to denote the values $E_0(n)$ and $E_1(n)$, respectively, when the sequential test S is applied.

2.2. *Efficiency of a sequential test.* As in the current test procedure, errors of two kinds may be committed in sequential analysis. We may reject H_0 when it is true (error of the first kind), or we may accept H_0 when H_1 is true (error of the second kind). With any sequential test there will be associated two numbers α and β between 0 and 1 such that if H_0 is true the probability is α that we shall commit an error of the first kind and if H_1 is true, the probability is β that we shall commit an error of the second kind. We shall say that two sequential tests S and S' are of equal strength if the values α and β associated with S are equal to the corresponding values α' and β' associated with S'. If $\alpha < \alpha'$ and $\beta \leq \beta'$, or if $\alpha \leq \alpha'$ and $\beta < \beta'$, we shall say that S is stronger than S'(S' is weaker than S). If $\alpha > \alpha'$ and $\beta < \beta'$, or if $\alpha < \alpha'$ and $\beta > \beta'$, we shall say that the strength of S is not comparable with that of S'.

Restricting ourselves to sequential tests of a given strength, we want to make the number of observations necessary for reaching a final decision as small as possible. If S and S' are two sequential tests of equal strength we shall say that S' is better than S if either $E_0(n \mid S') < E_0(n \mid S)$ and $E_1(n \mid S') \leq E_1(n \mid S)$, or $E_0(n \mid S') \leq E_0(n \mid S)$ and $E_1(n \mid S') < E_1(n \mid S)$. A sequential test will be said to be an admissible test if no better test of equal strength exists.

If a sequential test S satisfies both inequalities $E_0(n \mid S) \leq E_0(n \mid S')$ and $E_1(n \mid S) \leq E_1(n \mid S')$ for any sequential test S' of strength equal to that of S, then the test S can be considered to be a best sequential test. That such tests exist, i.e., that it is possible to minimize $E_0(n)$ and $E_1(n)$ simultaneously, is not proved here; but it is shown later (section 4.7) that for the so called sequential probability ratio test defined in section 3.1 both $E_0(n)$ and $E_1(n)$ are very nearly minimized.[4] Thus, for all practical purposes the sequential probability ratio test can be considered best.

Since it is unknown that a sequential test always exists for which both $E_0(n)$ and $E_1(n)$ are exactly minimized, we need a substitute definition of an optimum test. Several substitute definitions are possible. We could, for example, require that the test be admissible and the maximum of the two values $E_0(n)$ and $E_1(n)$ be minimized, or that the mean $\frac{E_0(n) + E_1(n)}{2}$, or some other weighted average be minimized All these definitions are equivalent if a sequential test exists for which both $E_0(n)$ and $E_1(n)$ are minimized; but if they cannot be minimized simultaneously the definitions differ. Which of them is chosen is of no significance for the purpose of this paper, since for the sequential probability ratio test proposed later both expected values $E_0(n)$ and $E_1(n)$ are, if not exactly, very nearly minimized. If we had a priori knowledge as to how frequently H_0 and how frequently H_1 will be true in the long run, it would be most reasonable to minimize a weighted average (weighted by the frequencies of H_0 and H_1, respectively) of $E_0(n)$ and $E_1(n)$. However, when such knowledge is absent, as is usually the case in practical applications, it is perhaps more reasonable to minimize the maximum of $E_0(n)$ and $E_1(n)$ than to minimize some weighted average of $E_0(n)$ and $E_1(n)$. Hence the following definition is introduced.

A sequential test S is said to be an optimum test if S is admissible and Max $[E_0(n \mid S), E_1(n \mid S)] \leq$ Max $[E_0(n \mid S'), E_1(n \mid S')]$ for all sequential tests S' of strength equal to that of S.

By the efficiency of a sequential test S is meant the value of the ratio[5]

$$\frac{\text{Max } [E_0(n \mid S^*), E_1(n \mid S^*)]}{\text{Max } [E_0(n \mid S), E_1(n \mid S)]}$$

where S^* is an optimum sequential test of strength equal to that of S.

2.3. *Efficiency of the current procedure, viewed as a particular case of a sequential test.* The current test procedure can be considered as a particular case of a sequential test. In fact, let N be the size of the sample used in the current procedure and let W_N be the critical region on which the test is based. Then the

[4] The author conjectures that $E_0(n)$ and $E_1(n)$ are exactly minimized for the sequential probability ratio test, but he did not succeed in proving this, except for a special class of problems (see section 4.7).

[5] The existence of an optimum sequential test is not essential for the definition of efficiency, since Max $[E_0(n \mid S^*), E_1(n \mid S^*)]$ could be replaced by the greatest lower bound of Max $[E(_0n \mid S'), E_1(n \mid S')]$ with respect to all sequential tests S' of strength equal to that of S

current procedure can be considered as a sequential test defined as follows: For all $m < N$, the regions R_m^0, R_m^1 are the empty subsets of the m-dimensional sample space M_m, and $R_m = M_m$. For $m = N$, R_N^1 is equal to W_N, R_N^0 is equal to the complement $\overline{W}_N$ of W_N and R_N is the empty set. Thus, for the current procedure we have $E_0(n) = E_1(n) = N$.

It will be seen later that the efficiency of the current test based on the most powerful critical region is rather low. Frequently it is below $\frac{1}{2}$. In other words, an optimum sequential test can attain the same α and β as the current most powerful test on the basis of an expected number of observations much smaller than the fixed number of observations needed for the current most powerful test.

In the next section we shall propose a simple sequential test procedure, called the sequential probability ratio test, which for all practical purposes can be considered an optimum sequential test. It will be seen that these sequential tests usually lead to average savings of about 50% in the number of trials as compared with the current most powerful test.

3. Sequential Probability Ratio Test

3.1. *Definition of the sequential probability ratio test.* We have seen in section 2 1 that the sequential test procedure is defined by subdividing the m-dimensional sample space M_m ($m = 1, 2, \cdots$, ad inf) into three mutually exclusive parts R_m^0, R_m^1 and R_m. The sequential process is terminated at the smallest value n of m for which the sample point lies either in R_n^0 or in R_n^1. If the sample point lies in R_n^0 we accept H_0 and if it lies in R_n^1 we accept H_1.

An indication as to the proper choice of the regions R_m^0, R_m^1 and R_m can be obtained from the following considerations: Suppose that before the sample is drawn there exists an a priori probability that H_0 is true and the value of this probability is known. Denote this a priori probability by g_0. Then the a priori probability that H_1 is true is given by $g_1 = 1 - g_0$, since it is assumed that the hypotheses H_0 and H_1 exhaust all possibilities. After a number of observations have been made we gain additional information which will affect the probability that H_i ($i = 0, 1$) is true. Let g_{0m} be the a posteriori probability that H_0 is true and g_{1m} the a posteriori probability that H_1 is true after m observations have been made. Then according to the well known formula of Bayes we have

$$g_{0m} = \frac{g_0 p_{0m}(x_1, \cdots, x_m)}{g_0 p_{0m}(x_1, \cdots, x_m) + g_1 p_{1m}(x_1, \cdots, x_m)} \tag{3.1}$$

and

$$g_{1m} = \frac{g_1 p_{1m}(x_1, \cdots, x_m)}{g_0 p_{0m}(x_1, \cdots, x_m) + g_1 p_{1m}(x_1, \cdots, x_m)} \tag{3.2}$$

where $p_{im}(x_1, \cdots, x_m)$ denotes the probability density in the m-dimensional sample space calculated under the hypothesis H_i ($i = 0, 1$).[6] As an abbreviation for $p_{im}(x_1, \cdots, x_m)$ we shall use simply p_{im}.

[6] If the probability distribution is discrete $p_{im}(x_1, \cdots, x_m)$ denotes the probability that the sample point $(x_1, \cdots, x_m)$ will be obtained.

Let d_0 and d_1 be two positive numbers less than 1 and greater than $\frac{1}{2}$. Suppose that we want to construct a sequential test such that the conditional probability of a correct decision under the condition that H_0 is accepted is greater than or equal to d_0, and the conditional probability of a correct decision under the condition that H_1 is accepted is greater than or equal to d_1.[7] Then the following sequential process seems reasonable: At each stage calculate g_{0m} and g_{1m}. If $g_{1m} \geq d_1$, accept H_1. If $g_{0m} \geq d_0$, accept H_0. If $g_{1m} < d_1$ and $g_{0m} < d_0$, draw an additional observation. R_m^0 in this sequential process is thus defined by the inequality $g_{0m} \geq d_0$, R_m^1 by the inequality $g_{1m} \geq d_1$, and R_m by the simultaneous inequalities $g_{1m} < d_1$ and $g_{0m} < d_0$. It is necessary that the sets R_m^0, R_m^1 and R_m be mutually exclusive and exhaustive. For this it suffices that the inequalities

$$g_{1m} = \frac{g_1 p_{1m}}{g_0 p_{0m} + g_1 p_{1m}} \geq d_1 \tag{3.3}$$

and

$$g_{0m} = \frac{g_0 p_{0m}}{g_0 p_{0m} + g_1 p_{1m}} \geq d_0 \tag{3.4}$$

be not fulfilled simultaneously. To show that (3 3) and (3.4) are incompatible, we shall assume that they are simultaneously fulfilled and derive a contradiction from this assumption. The two inequalities sum to

$$g_{1m} + g_{0m} \geq d_1 + d_0 . \tag{3.5}$$

Since $g_{0m} + g_{1m} = 1$, we have

$$1 \geq d_1 + d_0$$

which is impossible, since by assumption $d_i > \frac{1}{2}$ $(i = 0, 1)$. Hence it is proved that the sets R_m^0, R_m^1 and R_m are mutually exclusive and exhaustive.

The inequalities (3.3) and (3.4) are equivalent to the following inequalities, respectively:

$$\frac{p_{1m}}{p_{0m}} \geq \frac{g_0}{g_1} \frac{d_1}{1 - d_1} \tag{3.6}$$

and

$$\frac{p_{1m}}{p_{0m}} \leq \frac{g_0}{g_1} \frac{1 - d_0}{d_0}. \tag{3.7}$$

The constants on the right hand sides of (3.6) and (3.7) do not depend on m.

If an a priori probability of H_0 does not exist, or if it is unknown, the inequalities (3.6) and (3 7) suggest the use of the following sequential test: At each stage

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[7] The restriction $d_0 > 1/2$ and $d_1 > 1/2$ are imposed because otherwise it might happen that the hypothesis with the smaller a posteriori probability will be accepted.

calculate p_{1m}/p_{0m}. If $p_{1m} = p_{0m} = 0$, the value of the ratio p_{1m}/p_{0m} is defined to be equal to 1. Accept H_1 if

$$\frac{p_{1m}}{p_{0m}} \geq A. \tag{3.8}$$

Accept H_0 if

$$\frac{p_{1m}}{p_{0m}} \leq B. \tag{3.9}$$

Take an additional observation if

$$B < \frac{p_{1m}}{p_{0m}} < A. \tag{3.10}$$

Thus, the number n of observations required by the test is the smallest integral value of m for which either (3.8) or (3.9) holds. The constants A and B are chosen so that $0 < B < A$ and the sequential test has the desired value α of the probability of an error of the first kind and the desired value β of the probability of an error of the second kind. We shall call the test procedure defined by (3.8), (3.9) and (3.10), a sequential probability ratio test.

The sequential test procedure given by (3.8), (3.9) and (3.10) has been justified here merely on an intuitive basis. Section 4.7, however, shows that for this sequential test the expected values $E_0(n)$ and $E_1(n)$ are very nearly minimized.[8] Thus, for practical purposes this test can be considered an optimum test.

3.2. *Fundamental relations among the quantities α, β, A and B.* In this section the quantities α, β, A and B will be related by certain inequalities which are of basic importance for the sequential analysis.

Let $\{x_m\}$ $(m = 1, 2, \cdots, \text{ad inf.})$ be an infinite sequence of observations. The set of all possible infinite sequences $\{x_m\}$ is called the infinite dimensional sample space. It will be denoted by M_∞. Any particular infinite sequence $\{x_m\}$ is called a point of M_∞. For any set of n given real numbers $a_1, \cdots, a_n$ we shall denote by $C(a_1, \cdots, a_n)$ the subset of M_∞ which consists of all points (infinite sequences) $\{x_m\}$ $(m = 1, 2, \cdots, \text{ad inf.})$ for which $x_1 = a_1, \cdots, x_n = a_n$. For any values $a_1, \cdots, a_n$ the set $C(a_1, \cdots, a_n)$ will be called a cylindric point of order n. A subset S of M_∞ will be called a cylindric point, if there exists a positive integer n for which S is a cylindric point of order n. Thus, a cylindric point may be a cylindric point of order 1, or of order 2, etc. A cylindric point $C(a_1, \cdots, a_n)$ will be said to be of type 1 if

$$\frac{p_{1n}}{p_{0n}} = \frac{f_1(a_1)f_1(a_2)\cdots f_1(a_n)}{f_0(a_1)f_0(a_2)\cdots f_0(a_n)} \geq A$$

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[8] It seems likely to the author that $E_0(n)$ and $E_1(n)$ are exactly minimized for the sequential probability ratio test. However, he did not succeed in proving it, except for a special class of problems (see section 4.7).

and

$$B < \frac{p_{1m}}{p_{0m}} = \frac{f_1(a_1) \cdots f_1(a_m)}{f_0(a_1) \cdots f_0(a_m)} < A \qquad (m = 1, \cdots, n-1)$$

A cylindric point $C(a_1, \cdots, a_n)$ will be said to be of type 0 if

$$\frac{p_{1n}}{p_{0n}} = \frac{f_1(a_1) \cdots f_1(a_n)}{f_0(a_1) \cdots f_0(a_n)} \leq B$$

and

$$B < \frac{p_{1m}}{p_{0m}} = \frac{f_1(a_1) \cdots f_1(a_m)}{f_0(a_1) \cdots f_0(a_m)} < A \quad (m = 1, \cdots, n-1).$$

Thus, if a sample $(x_1, \cdots, x_n)$ is observed for which $C(x_1, \cdots, x_n)$ is a cylindric point of type i, the sequential test defined by (3.8), (3.9) and (3.10) leads to the acceptance of H_i $(i = 0, 1)$.

Let Q_i be the sum of all cylindric points of type i $(i = 0, 1)$. For any subset M of M_∞ we shall denote by $P_i(M)$ the probability of M calculated under the assumption that H_i is true $(i = 0, 1)$. Now we shall prove that

$$P_i(Q_0 + Q_1) = 1 \qquad (i = 0, 1) \tag{3.11}$$

This equation means that the probability is equal to one that the sequential process will eventually terminate. To prove (3 11) we shall denote the variate $\log \frac{f_1(x_i)}{f_0(x_i)}$ by z_i and $z_1 + \cdots + z_m$ by Z_m $(i, m = 1, 2, \cdots, \text{ad inf.})$. Furthermore, denote by n the smallest integer for which either $Z_n \geq \log A$ or $Z_n \leq \log B$. If no such finite integer n exists we shall say that $n = \infty$. Clearly, n is the number of observations required by the sequential test and (3.11) is proved if we show that the probability that $n = \infty$ is zero. But the latter statement was proved by the author elsewhere (see Lemma 1 in [4]). Hence equation (3.11) is proved.

With the help of (3.11) we shall be able to derive some important inequalities satisfied by the quantities α, β, A and B. Since for each sample $(x_1, \cdots, x_n)$ for which $C(x_1, \cdots, x_n)$ is an element of Q_1 the inequality $p_{1n}/p_{0n} \geq A$ holds, we see that

$$P_1(Q_1) \geq AP_0(Q_1) \tag{3.12}$$

Similarly, for each sample $(x_1, \cdots, x_n)$ for which $C(x_1, \cdots, x_n)$ is a point of Q_0 the inequality $p_{1n}/p_{0n} \leq B$ holds. Hence

$$P_1(Q_0) \leq BP_0(Q_0). \tag{3.13}$$

But $P_0(Q_1)$ is the probability of committing an error of the first kind and $P_1(Q_0)$ is the probability of making an error of the second kind. Thus, we have

$$P_0(Q_1) = \alpha, \qquad P_1(Q_0) = \beta. \tag{3 14}$$

Since Q_0 and Q_1 are disjoint, it follows from (3.11) that

$$P_0(Q_0) = 1 - \alpha; \qquad P_1(Q_1) = 1 - \beta. \tag{3.15}$$

From the relations (3.12)–(3.15) we obtain the important inequalities

$$1 - \beta \geq A\,\alpha \tag{3.16}$$

and

$$\beta \leq B\,(1 - \alpha). \tag{3.17}$$

These inequalities can be written as

$$\frac{\alpha}{1 - \beta} \leq \frac{1}{A} \tag{3.18}$$

and

$$\frac{\beta}{1 - \alpha} \leq B. \tag{3.19}$$

The above inequalities are of great value in practical applications, since they supply upper limits for α and β when A and B are given. For instance, it follows immediately from (3.18) and (3.19), and the fact that $0 < \alpha < 1, 0 < \beta < 1$ that

$$\alpha \leq \frac{1}{A} \tag{3.20}$$

and

$$\beta \leq B. \tag{3.21}$$

A pair of values α and β can be represented by a point in the plane with the coordinates α and β. It is of interest to determine the set of all points (α, β) which satisfy the inequalities (3.18) and (3 19) for given values of A and B. Consider the straight lines L_1 and L_2 in the plane given by the equations

$$A\alpha = 1 - \beta \tag{3 22}$$

and

$$\beta = B(1 - \alpha), \tag{3.23}$$

respectively. The line L_1 intersects the abscissa axis at $\alpha = \frac{1}{A}$ and the ordinate axis at $\beta = 1$. The line L_2 intersects the abscissa axis at $\alpha = 1$ and the ordinate axis at $\beta = B$. The set of all points (α, β) which satisfy the inequalities (3.18) and (3.19) is the interior and the boundary of the quadrilateral determined by the lines L_1, L_2 and the coordinate axes. This set is represented by the shaded area in figure 1

The fundamental inequalities (3 18) and (3.19) were derived under the assumption that $x_1, x_2, \cdots$, ad inf are independent observations on the same random

variable X. The independence of the observations is, however, not necessary for the validity of (3.18) and (3.19). In fact, the independence of the observations was used merely to show the validity of (3.11). But (3.11) can be shown to hold also for dependent observations under very general conditions. Hence, if H_i states that the joint distribution of $x_1, x_2, \cdots, x_m$ is given by the joint probability density function $p_{im}(x_1, \cdots, x_m)$[9] ($i = 0, 1$; $m = 1, 2, \cdots$, ad inf.) and if (3.11) holds, then for the sequential test of H_0 against H_1, as defined by (3.8), (3.9) and (3.10), the inequalities (3.18) and (3.19) remain valid. For instance, let λ_0 and λ_1 be two different positive values <1 and let $H_i (i = 0, 1)$ be the hypothesis that the joint probability density function of $x_1, \cdots, x_m$ is given by

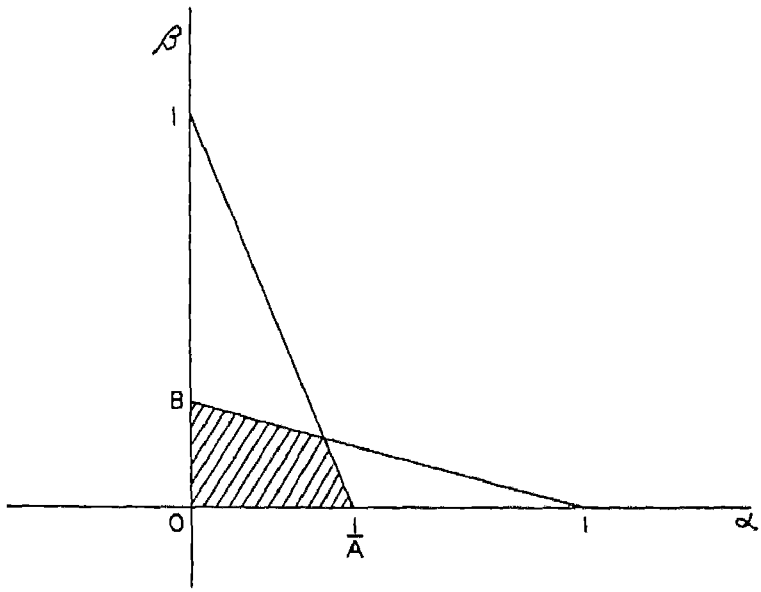

FIG. 1

$$p_{im}(x_1, \cdots, x_m = \frac{1}{(2\pi)^{m/2}} e^{-\frac{1}{2}x_1^2 - \frac{1}{2}\sum_{j=2}^{m}(x_j - \lambda_i x_{j-1})^2} \qquad (i = 0, 1)$$

i.e., that x_1 and $(x_j - \lambda_i x_{j-1}) (j = 2, 3, \cdots$, ad inf.) are normally and independently distributed with zero means and unit variances, then the inequalities (3.18) and (3.19) will hold for the sequential test defined by (3.8), (3.9) and (3.10).

3.3. *Determination of the values A and B in practice.* Suppose that we wish to have a sequential test such that the probability of an error of the first kind is equal to α and the probability of an error of the second kind is equal to β. De-

[9] Of course, for any positive integers m and m' with $m < m'$ the marginal distribution of $x_1, \cdots, x_m$ determined on the basis of the joint distribution $P_{im'}(x_1, \cdots, x_{m'})$ must be equal to $P_{im}(x_1, \cdots, x_m)$.

note by $A(\alpha, \beta)$ and $B(\alpha, \beta)$ the values of A and B for which the probabilities of the errors of the first and second kinds will take the desired values α and β The exact determination of the values $A(\alpha, \beta)$ and $B(\alpha, \beta)$ is rather laborious, as will be seen in Section 3.4. The inequalities at our disposal, however, permit the problem to be solved satisfactorily for practical purposes. From (3.18) and (3.19) it follows that

$$A(\alpha, \beta) \leq \frac{1-\beta}{\alpha} \tag{3.24}$$

and

$$B(\alpha, \beta) \geq \frac{\beta}{1-\alpha}. \tag{3.25}$$

Suppose we put $A = \dfrac{1-\beta}{\alpha} = a(\alpha, \beta)$ (say), and $B = \dfrac{\beta}{1-\alpha} = b(\alpha, \beta)$ (say). Then A is greater than or equal to the exact value $A(\alpha, \beta)$, and B is less than or equal to the exact value $B(\alpha, \beta)$. This procedure, of course, changes the probabilities of errors of the first and second kind. If we were to use the exact value of B and a value of A which is greater than the exact value, then evidently we would lower the value of α, but slightly increase the value of β. Similarly, if we were to use the exact value of A and a value of B which is below the exact value, then we would lower the value of β, but slightly increase the value of α. Thus, it is not clear what will be the resulting effect on α and β if a value of A is used which is higher than the exact value, and a value of B is used which is lower than the exact value. Denote by α' and β' the resulting probabilities of errors of the first and second kind, respectively, if we put $A = \dfrac{1-\beta}{\alpha}$ and $B = \dfrac{\beta}{1-\alpha}$.

We now derive inequalities satisfied by the quantities α', β', α and β. Substituting $a(\alpha, \beta)$ for A, $b(\alpha, \beta)$ for B, α' for α and β' for β we obtain from (3.18) and (3.19)

$$\frac{\alpha'}{1-\beta'} \leq \frac{1}{a(\alpha, \beta)} = \frac{\alpha}{1-\beta} \tag{3.26}$$

and

$$\frac{\beta'}{1-\alpha'} \leq b(\alpha, \beta) = \frac{\beta}{1-\alpha}. \tag{3.27}$$

From these inequalities it follows that

$$\alpha' \leq \frac{\alpha}{1-\beta} \tag{3.28}$$

and

$$\beta' \leq \frac{\beta}{1-\alpha}. \tag{3.29}$$

Multiplying (3 26) by $(1 - \beta)(1 - \beta')$ and (3.27) by $(1 - \alpha)(1 - \alpha')$ and adding the two resulting inequalities, we have

$$\alpha' + \beta' \leq \alpha + \beta. \tag{3.30}$$

Thus, we see that at least one of the inequalities $\alpha' \leq \alpha$ and $\beta' \leq \beta$ must hold. In other words, by using $a(\alpha, \beta)$ and $b(\alpha, \beta)$ instead of $A(\alpha, \beta)$ and $B(\alpha, \beta)$, respectively, at most one of the probabilities α and β may be increased.

If α and β are small (say less than 05), as they frequently will be in practical applications, $\frac{\alpha}{1 - \beta}$ and $\frac{\beta}{1 - \alpha}$ are nearly equal to α and β, respectively. Thus, we see from (3 28) and (3.29) that the quantity by which α' can possibly exceed α, or β' can exceed β, must be small. Section 3.4 contains further inequalities which show that the amount by which $\alpha'(\beta')$ can possibly exceed $\alpha(\beta)$ is indeed extremely small. Thus, for all practical purposes $\alpha' \leq \alpha$ and $\beta' \leq \beta$.

If $f_1(x)$ (the distribution under the alternative hypothesis) is sufficiently near $f_0(x)$ (the distribution under the null hypothesis), $A(\alpha, \beta)$ and $B(\alpha, \beta)$ will be nearly equal to $\frac{1 - \beta}{\alpha}$ and $\frac{\beta}{1 - \alpha}$, respectively; and consequently α' and β' are also very nearly equal to α and β respectively. The reason that (3.18) and (3.19) and therefore also (3.24) and (3 25), are inequalities instead of equalities is that the sequential process may terminate with $\frac{p_{1n}}{p_{0n}} > A$ or $\frac{p_{1n}}{p_{0n}} < B$. If at the final stage $\frac{P_{1n}}{P_{0n}}$ were exactly equal to A or B, then $A(\alpha, \beta)$ and $B(\alpha, \beta)$ would be exactly $\frac{1 - \beta}{\alpha}$ and $\frac{\beta}{1 - \alpha}$, respectively. If $f_1(x)$ is near $f_0(x)$, it is almost certain that the value of $\frac{P_{1n}}{P_{0n}}$ is changed only slightly by one additional observation. Thus, at the final stage $\frac{P_{1n}}{P_{0n}}$ will be only slightly above A, or slightly below B and consequently $A(\alpha, \beta)$ and $B(\alpha, \beta)$ will be nearly equal to $\frac{1 - \beta}{\alpha}$ and $\frac{\beta}{1 - \alpha}$, respectively If fractional observations were possible, that is to say, if the number of observations were a continuous variable, $\frac{P_{1m}}{P_{0m}}$ would also be a continuous function of m and consequently $A(\alpha, \beta)$ and $B(\alpha, \beta)$ would be exactly equal to $\frac{1 - \beta}{\alpha}$ and $\frac{\beta}{1 - \alpha}$, respectively. Thus, we have inequalities in (3.24) and (3.25) instead of equalities merely on account of the fact that the number m of observations is discontinuous, i.e., m can take only integral values

Hence for all practical purposes the following procedure can be adopted: *To construct a sequential test such that the probability of an error of the first kind does not exceed α and the probability of an error of the second kind does not exceed β, put*

$A = \frac{1-\beta}{\alpha}$ *and* $B = \frac{\beta}{1-\alpha}$ *and carry out the sequential test as defined by the inequalities* (3.8), (3.9) *and* (3.10).

In most practical cases the calculation of the exact values $A(\alpha, \beta)$ and $B(\alpha, \beta)$ will be of little interest for the following reasons: When $A = a(\alpha, \beta) = \frac{1-\beta}{\alpha}$ and $B = b\,(\alpha, \beta) = \frac{\beta}{1-\alpha}$, the probability α' of an error of the first kind cannot exceed α and the probability β' of an error of the second kind cannot exceed β, except by a very small quantity which can be neglected for practical purposes. Thus, for all practical purposes the use of $a(\alpha, \beta)$ and $b(\alpha, \beta)$ instead of $A(\alpha, \beta)$ and $B(\alpha, \beta)$ will not decrease the strength of the sequential test. The only possible disadvantage from the substitution is that it may increase the expected number of trials necessary for a decision. Since the discrepancy between $A(\alpha, \beta)$ and $B(\alpha, \beta)$ on the one hand and $a(\alpha, \beta)$ and $b(\alpha, \beta)$ on the other, arises only from the discontinuity of the number m of observations, it is clear that the increase in the expected number of trials caused by the use of $a(\alpha, \beta)$ and $b(\alpha, \beta)$ will be slight. This slight increase, however, cannot be considered entirely a loss for the following reason: if $a(\alpha, \beta) > A(\alpha, \beta)$ or $b(\alpha, \beta) < B(\alpha, \beta)$, then we can sharpen the inequality (3.30) to $\alpha' + \beta' < \alpha + \beta$. Hence by using $a(\alpha, \beta)$ and $b(\alpha, \beta)$ we gain in strength.

The fact that for practical purposes we may put $A = a(\alpha, \beta)$ and $B = b(\alpha, \beta)$ brings out a surprising feature of the sequential test as compared with current tests. While current tests cannot be carried out without finding the probability distribution of the statistic on which the test is based, there are no distribution problems in connection with sequential tests. In fact, $a(\alpha, \beta)$ and $b(\alpha, \beta)$ depend on α and β only, and the ratio $\frac{p_{1m}}{p_{0m}}$ can be calculated from the data of the problem without solving any distribution problems. Distribution problems arise in connection with the sequential process only if it is desired to find the probability distribution of the number of trials necessary for reaching a final decision (This subject is discussed later.) But this is of secondary importance as long as we know that the sequential test on the average leads to a saving in the number of trials.

3.4. *Probability of accepting H_0 (or H_1) when some third hypothesis H is true.* In Section 3 2 we were concerned with the probability that the sequential probability ratio test will lead to the acceptance of H_0 (or H_1) when H_0 or H_1 is true. Since in Part II we shall admit an infinite set of alternatives, and since this is the practically important case, it is of interest to study the probability of accepting H_0 (or H_1) when any third hypothesis H, not necessarily equal to H_0 or H_1, is true. Let H be the hypothesis that the distribution of X is given by $f(x)$. If $f(x)$ is equal to $f_0(x)$ or $f_1(x)$ we have the special case discussed in Section 3.2. In what follows in this and the subsequent sections any probability relationship

will be stated on the assumption that H is true, unless a statement to the contrary is explicitly made. Denote by γ the probability that the sequential probability ratio test will lead to the acceptance of H_1.[10] Clearly, if $H = H_0$, then $\gamma = \alpha$ and if $H = H_1$, then $\gamma = 1 - \beta$.

The probability γ can readily be derived on the basis of the general theory of cumulative sums given in [4]. Denote $\log \dfrac{f_1(x_i)}{f_0(x_i)}$ by z_i. Then $\{z_i\}$ ($i = 2, \cdots$, ad inf.) is a sequence of independent random variables each having the same distribution. Denote by Z_j the sum of the first j elements of the sequence $\{z_i\}$ i.e.,

$$Z_j = z_1 + \cdots + z_j \qquad (j = 1, 2, \cdots, \text{ad inf.}) \tag{3.31}$$

For any relation R we shall denote by $P(R)$ the probability that R holds. For any random variable Y the symbol EY will denote the expected value of Y. Let n be the smallest positive integer for which either $Z_n \geq \log A$ or $Z_n \leq \log B$ holds. If $\log B < Z_m < \log A$ holds for $m = 1, 2, \cdots$, ad inf., we shall say that $n = \infty$. Obviously, n is the number of observations required by the sequential probability ratio test. As we have seen in Section 3.3, in practice we shall put $A = a(\alpha, \beta) = \dfrac{1 - \beta}{\alpha}$ and $B = b(\alpha, \beta) = \dfrac{\beta}{1 - \alpha}$. Since B must be less than A, we shall consider only values α and β for which $\dfrac{1 - \beta}{\alpha} > \dfrac{\beta}{1 - \alpha}$. This inequality is equivalent to $\alpha + \beta < 1$, which in turn implies that $B < 1$ and $A > 1$. Thus, in all that follows it will be assumed that $A > 1$ and $B < 1$. We shall also assume that the variance of z_i is not zero.

According to Lemma 1 in [4] the relation $P(n = \infty) = 0$ holds. Hence, the probability is equal to one that the sequential process will eventually terminate. This implies that the probability of accepting H_0 is equal to $1 - \gamma$.

Let z be a random variable whose distribution is equal to the common distribution of the variates z_i ($i = 1, 2, \cdots$, ad inf.). Denote by $\varphi(t)$ the moment generating function of z, i.e.,

$$\varphi(t) = Ee^{zt}.$$

It was shown in [4] that under very mild restrictions on the distribution of z there exists exactly one real value h such that $h \neq 0$ and $\varphi(h) = 1$. Furthermore, it was shown in [4] (see equation (16) in [4]) that

$$Ee^{Z_n h} = 1. \tag{3.32}$$

Let E^* be the conditional expected value of $e^{Z_n h}$ under the restriction that H_0 is accepted, i.e., that $Z_n \leq \log B$, and let E^{**} be the conditional expected value of $e^{Z_n h}$ under the restriction that H_1 is accepted, i.e., that $Z_n \geq \log A$. Then we obtain from (3.32)

$$(1 - \gamma)E^* + \gamma E^{**} = 1 \tag{3.33}$$

---

[10] The probability that H_0 will be accepted is equal to $1 - \gamma$, as will be seen later.

Solving for γ we obtain

$$\gamma = \frac{1 - E^*}{E^{**} - E^*}. \tag{3.34}$$

If both the absolute value of Ez and the variance of z are small, which will be the case when $f_1(x)$ is near $f_0(x)$, E^* and E^{**} will be nearly equal to B^h and A^h, respectively. Hence, in this case a good approximation to γ is given by the expression

$$\bar{\gamma} = \frac{1 - B^h}{A^h - B^h}. \tag{3.35}$$

It is easy to verify that $h = 1$ if $H = H_0$, and $h = -1$ if $H = H_1$. The difference $\bar{\gamma} - \gamma$ approaches zero if both the mean and the variance of z converge to zero.

To judge the goodness of the approximation given by $\bar{\gamma}$, it is desirable to derive lower and upper limits for γ. Such limits for γ can be obtained by deriving lower and upper limits for E^* and E^{**}. First we consider the case when $h > 0$. Let ζ be a real variable restricted to values >1, and let ρ be a positive variable restricted to values <1. For any random variable Y and any relationship R we shall denote by $E(Y \mid R)$ the conditional expected value of Y under the restriction that R holds. It was shown in [4] that the following inequalities hold:[11]

$$B^h \left\{ \underset{\zeta}{\text{g.l.b.}}\ \zeta E\left(e^{hz} \mid e^{hz} \leq \frac{1}{\zeta}\right)\right\} \leq E^* \leq B^h \qquad (h > 0) \tag{3.36}$$

and

$$A^h \leq E^{**} \leq A^h \left\{ \underset{\rho}{\text{l.u.b.}}\ \rho E\left(e^{hz} \mid e^{hz} \geq \frac{1}{\rho}\right)\right\} \qquad (h > 0). \tag{3.37}$$

The symbol $\underset{\zeta}{\text{g.l.b.}}$ stands for the greatest lower bound with respect to ζ, and the symbol $\underset{\rho}{\text{l.u.b.}}$ stands for least upper bound with respect to ρ. Putting

$$\underset{\zeta}{\text{g.l.b.}}\ \zeta E\left(e^{hz} \mid e^{hz} \leq \frac{1}{\zeta}\right) = \eta \tag{3.38}$$

and

$$\underset{\rho}{\text{l.u.b.}}\ \rho E\left(e^{hz} \mid e^{hz} \geq \frac{1}{\rho}\right) = \delta, \tag{3.39}$$

the inequalities (3.36) and (3.37) can be written as

$$B^h \eta \leq E^* \leq B^h \qquad (h > 0) \tag{3.40}$$

---

[11] See relations (23) and (26) in [4]. The notation used here is somewhat different from that in [4].

and

$$A^h \leq E^{**} \leq A^h \delta \qquad (h > 0). \tag{3.41}$$

Since $B < 1$ and $A > 1$, we see that $E^* < 1$ and $E^{**} > 1$ if $h > 0$. From this and the relations (3.34), (3.40) and (3.41) it follows easily that

$$\frac{1 - B^h}{\delta A^h - B^h} \leq \gamma \leq \frac{1 - \eta B^h}{A^h - \eta B^h} \qquad (h > 0) \tag{3.42}$$

If $h < 0$, limits for γ can be obtained as follows: Let $z' = -z$, $A' = \frac{1}{B}$, $B' = \frac{1}{A}$. Then $h' = -h > 0$ and $\gamma' = 1 - \gamma$. Thus, according to (3.42) we have

$$\frac{1 - (B')^{h'}}{\delta'(A')^{h'} - (B')^{h'}} \leq \gamma' \leq \frac{1 - \eta'(B')^{h'}}{(A')^{h'} - \eta'(B')^{h'}} \tag{3.43}$$

where δ' and η' are equal to the expressions we obtain from (3.38) and (3.39), respectively, by substituting h' for h and z' for z. Since η and δ depend only on the product $hz = h'z'$, we see that $\delta' = \delta$ and $\eta' = \eta$. Hence, we obtain from (3.43)

$$\frac{1 - A^h}{\delta B^h - A^h} \leq 1 - \gamma \leq \frac{1 - \eta A^h}{B^h - \eta A^h} \qquad (h < 0) \tag{3.44}$$

where δ and η are given by (3.38) and (3.39), respectively.

In Section 3.5 we shall calculate the value of η and δ for binomial and normal distributions. If the limits of γ, as given in (3.42) and (3.44), are too far apart, it may be desirable to determine the exact value of γ, or at least to find a closer approximation to γ than that given in (3.35). A solution of this problem is given in [4] (see section 7 of that paper). There the exact value of γ is derived when z can take only a finite number of integral multiples of a constant d. If z does not have this property, arbitrarily fine approximation to the value of γ can be obtained, since the distribution of z can be approximated to any desired degree by a discrete distribution of the type mentioned before if the constant d is chosen sufficiently small. The results obtained in [4] can be stated as follows: There is no loss of generality in assuming that $d = 1$, since the quantity d can be chosen as the unit of measurement. Thus, we shall assume that z takes only a finite number of integral values. Let g_1 and g_2 be two positive integers such that $P(z = -g_1)$ and $P(z = g_2)$ are positive and z can take only integral values $\geq -g_1$ and $\leq g_2$. Denote $P(z = i)$ by h_i. Then the moment generating function of z is given by

$$\varphi(t) = \sum_{i=-g_1}^{g_2} h_i e^{it}.$$

Put $u = e^t$ and let $u_1, \cdots u_g$ be the $g = g_1 + g_2$ roots of the equation of g-th degree

$$\sum_{i=-g_1}^{g_2} h_i u^i = 1. \tag{3.45}$$

Denote by $[a]$ the smallest integer $\geq \log A$, and by $[b]$ the largest integer $\leq \log B$. Then Z_n can take only the values

(3.46) $\quad [b] - g_1 + 1, [b] - g_1 + 2, \cdots, [b], [a], [a] + 1, \cdots, [a] + g_2 - 1.$

Denote the g different integers in (3.46) by $c_1, \cdots, c_g$, respectively. Let Δ be the determinant value of the matrix $\| u_i^{c_j} \|$ $(i, j = 1, \cdots, g)$ and let Δ_j be the determinant we obtain from Δ by substituting 1 for the elements in the j-th column. Then, if $\Delta \neq 0$, the probability that $Z_n = c_j$ is given by

$$P(Z_n = c_j) = \frac{\Delta_j}{\Delta}. \tag{3.47}$$

Hence

$$\gamma = P(Z_n \geq [a]) = \sum_j \frac{\Delta_j}{\Delta} \tag{3.48}$$

where the summation is to be taken over all vaues of j for which $c_j \geq [a]$.

3.5. *Calculation of δ and η for binomial and normal distributions.* Let X be a random variable which can take only the values 0 and 1. Let the probability that $X = 1$ be p_i if H_i is true $(i = 0, 1)$, and p if H is true. Denote $1 - p$ by q and $1 - p_i$ by q_i $(i = 0, 1)$. Then $f_i(1) = p_i$; $f_i(0) = q_i$, $f(1) = p$ and $f(0) = q$. It can be assumed without loss of generality that $p_1 > p_0$. The moment generating function of $z = \log \dfrac{f_1(x)}{f_0(x)}$ is given by

$$\varphi(t) = E\left(\frac{f_1(x)}{f_0(x)}\right)^t = p\left(\frac{p_1}{p_0}\right)^t + q\left(\frac{q_1}{q_0}\right)^t.$$

Let $h \neq 0$ be the value of t for which $\varphi(h) = 1$, i.e.,

$$p\left(\frac{p_1}{p_0}\right)^h + q\left(\frac{q_1}{q_0}\right)^h = 1.$$

First we consider the case when $h > 0$. It is clear that $e^{zh} = \left(\dfrac{f_1(x)}{f_0(x)}\right)^h > 1$ implies that $x = 1$. Hence $e^{zh} > 1$ implies that $e^{zh} = \left(\dfrac{f_1(1)}{f_0(1)}\right)^h = \left(\dfrac{p_1}{p_0}\right)^h$. From this and the definition of δ given in (3.39) it follows that

$$\delta = \left(\frac{p_1}{p_0}\right)^h \qquad (h > 0). \tag{3.49}$$

Similarly, the inequality $e^{zh} < 1$ implies that $e^{zh} = \left(\dfrac{q_1}{q_0}\right)^h$. From this and the definition of η given in (3.38) it follows that

$$\eta = \left(\frac{q_1}{q_0}\right)^h \qquad (h > 0). \tag{3.50}$$

If $h < 0$, it can be shown in a similar way that

$$\delta = \left(\frac{q_1}{q_0}\right)^h \qquad (h < 0) \tag{3.51}$$

and

$$\eta = \left(\frac{p_1}{p_0}\right)^h \qquad (h < 0). \tag{3.52}$$

Now we shall calculate the values of δ and η if X is normally distributed. Let

$$f_i(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta_i)^2} \qquad (i = 0, 1) \tag{3.53}$$

and

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2}. \tag{3.54}$$

We can assume without loss of generality that $\theta_0 = -\Delta$ and $\theta_1 = \Delta$ where $\Delta > 0$, since this can always be achieved by a translation. Then

$$z = \log \frac{f_1(x)}{f_0(x)} = 2\Delta x. \tag{3.55}$$

The moment generating function of z is given by

$$\varphi(t) = e^{2\Delta\theta t + 2\Delta^2 t^2}. \tag{3.56}$$

Hence

$$h = -\frac{\theta}{\Delta}. \tag{3.57}$$

Substituting this value of h in (3.38) and (3.39) we obtain

$$\delta = \underset{\rho}{\text{l.u.b.}}\ \rho E\left(e^{-2\theta x} \mid e^{-2\theta x} \geq \frac{1}{\rho}\right) \tag{3.58}$$

and

$$\eta = \underset{\zeta}{\text{g.l.b.}}\ \zeta E\left(e^{-2\theta x} \mid e^{-2\theta x} \leq \frac{1}{\zeta}\right). \tag{3.59}$$

For any relation R let $P^*(R)$ denote the probability that the relation R holds calculated under the assumption that the distribution of x is normal with mean θ and variance unity. Furthermore, let $P^{**}(R)$ denote the probability that R holds if the distribution of x is normal with mean $-\theta$ and variance unity. Since $e^{-2\theta x}$ is equal to the ratio of the normal probability density function with mean $-\theta$ and variance unity to the normal probability density function with mean θ and variance unity, we see that

$$E\left(e^{-2\theta x} \mid e^{-2\theta x} \geq \frac{1}{\rho}\right) = \frac{P^{**}\left(e^{-2\theta x} \geq \frac{1}{\rho}\right)}{P^{*}\left(e^{-2\theta x} \geq \frac{1}{\rho}\right)}, \tag{3.60}$$

and

$$E\left(e^{-2\theta x} \mid e^{-2\theta x} \leq \frac{1}{\zeta}\right) = \frac{P^{**}\left(e^{-2\theta x} \leq \frac{1}{\zeta}\right)}{P^{*}\left(e^{-2\theta x} \leq \frac{1}{\zeta}\right)}. \tag{3.61}$$

It can easily be verified that the right hand side expressions in (3.60) and (3.61) have the same values for $\theta = \lambda$ as for $\theta = -\lambda$. Thus, also δ and η have the same values for $\theta = \lambda$ as for $\theta = -\lambda$. It will be, therefore, sufficient to compute δ and η for negative values of θ. Let $\theta = -\lambda$ where $\lambda > 0$. First we show that $\eta = \frac{1}{\delta}$. Clearly

$$\frac{\zeta P^{**}\left(e^{2\lambda x} \leq \frac{1}{\zeta}\right)}{P^{*}\left(e^{2\lambda x} \leq \frac{1}{\zeta}\right)} = \frac{\zeta P^{**}(e^{-2\lambda x} \geq \zeta)}{P^{*}(e^{-2\lambda x} \geq \zeta)} \qquad (1 \leq \zeta < \infty). \tag{3.62}$$

Putting $\zeta = \frac{1}{\rho}$ $(0 < \rho \leq 1)$ in (3.62) gives

$$\frac{\zeta P^{**}\left(e^{2\lambda x} \leq \frac{1}{\zeta}\right)}{P^{*}\left(e^{2\lambda x} \leq \frac{1}{\zeta}\right)} = \frac{P^{**}\left(e^{-2\lambda x} \geq \frac{1}{\rho}\right)}{\rho P^{*}\left(e^{-2\lambda x} \geq \frac{1}{\rho}\right)}. \tag{3.63}$$

Hence

$$\eta = \underset{\zeta}{\text{g.l.b.}} \left\{\frac{\zeta P^{**}\left(e^{2\lambda x} \leq \frac{1}{\zeta}\right)}{P^{*}\left(e^{2\lambda x} \leq \frac{1}{\zeta}\right)}\right\} = \frac{1}{\underset{\rho}{\text{l.u.b.}} \left\{\frac{\rho P^{*}\left(e^{-2\lambda x} \geq \frac{1}{\rho}\right)}{P^{**}\left(e^{-2\lambda x} \geq \frac{1}{\rho}\right)}\right\}}. \tag{3.64}$$

Because of the symmetry of the normal distribution, it is easily seen that

$$\underset{\rho}{\text{l.u.b.}} \left\{\frac{\rho P^{*}\left(e^{-2\lambda x} \geq \frac{1}{\rho}\right)}{P^{**}\left(e^{-2\lambda x} \geq \frac{1}{\rho}\right)}\right\} = \underset{\rho}{\text{l.u.b.}} \left\{\frac{\rho P^{**}\left(e^{2\lambda x} \geq \frac{1}{\rho}\right)}{P^{*}\left(e^{2\lambda x} \geq \frac{1}{\rho}\right)}\right\} = \delta.$$

Hence

$$\eta = \frac{1}{\delta}. \tag{3.65}$$

Now we shall calculate the value of δ. Denote $\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}\,dt$ by $G(x)$. Then

$$P^{**}\left(e^{2\lambda x} \geq \frac{1}{\rho}\right) = P^{**}\left(2\lambda x \geq \log \frac{1}{\rho}\right)$$
$$= P^{**}\left(x \geq \frac{1}{2\lambda}\log \frac{1}{\rho}\right) = G\left(2\lambda \log \frac{1}{\rho} - \lambda\right).$$

Similarly

$$P^{*}\left(e^{2\lambda x} \geq \frac{1}{\rho}\right) = P^{*}\left(x \geq \frac{1}{2\lambda}\log \frac{1}{\rho}\right) = G\left(\frac{1}{2\lambda}\log \frac{1}{\rho} + \lambda\right).$$

Denote $\frac{1}{2\lambda}\log \frac{1}{\rho}$ by u. Since ρ can vary from 0 to 1, u can take any value from 0 to ∞. Since $\rho = e^{-2\lambda u}$, we have

$$\delta = \underset{\rho}{\text{l.u.b.}}\left\{\frac{\rho P^{**}\left(e^{2\lambda x} \geq \frac{1}{\rho}\right)}{P^{*}\left(e^{2\lambda x} \geq \frac{1}{\rho}\right)}\right\} = \underset{u}{\text{l.u.b.}}\left\{e^{-2\lambda u}\frac{G(u-\lambda)}{G(u+\lambda)}\right\} \quad (0 \leq u < \infty). \tag{3.66}$$

We shall prove that

$$e^{-2u\lambda}\frac{G(u-\lambda)}{G(u+\lambda)} = \chi(u) \qquad \text{(say)} \tag{3.67}$$

is a monotonically decreasing function of u and consequently the maximum is at $u = 0$. For this purpose it suffices to show that the derivative of $\log \chi(u)$ is never positive. Now

$$\log \chi(u) = \log G(u-\lambda) - \log G(u+\lambda) - 2\lambda u. \tag{3.68}$$

Denote $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$ by $\Phi(x)$. Since $\frac{d}{du}G(u) = -\Phi(u)$ it follows from (3.68) that

$$\frac{d}{du}\log \chi(u) = -\frac{\Phi(u-\lambda)}{G(u-\lambda)} + \frac{\Phi(u+\lambda)}{G(u+\lambda)} - 2\lambda. \tag{3.69}$$

It follows from the mean value theorem that the right hand side of (3.69) is never positive if $\frac{d}{du}\left(\frac{\Phi(u)}{G(u)}\right)$ is equal to or less than 1 for all values of u. Thus, we need merely to show that

$$\frac{d}{du}\left(\frac{\Phi(u)}{G(u)}\right) = \frac{\Phi'(u)G(u) - G'(u)\Phi(u)}{G^2(u)}$$

$$= \frac{\Phi'(u)G(u) + \Phi^2(u)}{G^2(u)} = \frac{\Phi^2(u)}{G^2(u)} - u\frac{\Phi(u)}{G(u)} \leq 1. \tag{3.70}$$

Denote $\frac{\Phi(u)}{G(u)}$ by y. The roots of the equation $y^2 - uy - 1 = 0$ are

$$y = \frac{u \pm \sqrt{u^2 + 4}}{2}.$$

Hence the inequality $y^2 - uy - 1 \leq 0$ holds if and only if

$$\frac{u - \sqrt{u^2 + 4}}{2} \leq y \leq \frac{u + \sqrt{u^2 + 4}}{2}.$$

Since y cannot be negative, this inequality is equivalent to

$$\frac{\Phi(u)}{G(u)} = y \leq \frac{u + \sqrt{u^2 + 4}}{2}. \tag{3.71}$$

Thus we have merely to prove (3.71). We shall show that (3.71) holds for all real values of u. Birnbaum has shown [5] that for $u > 0$

$$\frac{\sqrt{u^2 + 4} - u}{2}\,\Phi(u) \leq G(u). \tag{3.72}$$

Hence

$$\frac{\Phi(u)}{G(u)} \leq \frac{2}{\sqrt{u^2 + 4} - u} = \frac{\sqrt{u^2 + 4} + u}{2} \qquad (u > 0) \tag{3.73}$$

which proves (3.71) for $u > 0$. Now we prove (3.71) for $u < 0$. Let $u = -v$ where $v > 0$. Then it follows from (3.73) that

$$\frac{\Phi(v)}{G(v)} \leq \frac{2}{\sqrt{4 + v^2} - v} \tag{3.74}$$

Taking reciprocals, we obtain from (3.74)

$$\frac{G(v)}{\Phi(v)} \geq \frac{\sqrt{4 + v^2} - v}{2}. \tag{3.75}$$

Since

$$\frac{G(u)}{\Phi(u)} \geq \frac{G(v) + 2v\Phi(v)}{\Phi(v)} = \frac{G(v)}{\Phi(v)} + 2v$$

we obtain from (3.75)

$$\frac{G(u)}{\Phi(u)} \geq \frac{\sqrt{v^2 + 4} + 3v}{2} \geq \frac{\sqrt{v^2 + 4} + v}{2}. \tag{3.76}$$

Taking reciprocals, we obtain

$$\frac{\Phi(u)}{G(u)} \leq \frac{2}{\sqrt{v^2 + 4} + v} = \frac{\sqrt{v^2 + 4} - v}{2} = \frac{\sqrt{u^2 + 4} + u}{2}.$$

Hence (3.71) is proved for all values of u and consequently δ is equal to the value of the expression (3.67) if we substitute 0 for u. Thus,

$$\delta = \frac{G(-\lambda)}{G(\lambda)}. \tag{3.77}$$

4. The Number of Observations Required by the Sequential Probability Ratio Test

4.1. *Expected number of observations necessary for reaching a decision.* As before, let

$$z = \log \frac{f_1(x)}{f_0(x)}, \qquad z_i = \log \frac{f_1(x_i)}{f_0(x_i)} \qquad (i = 1, 2, \cdots, \text{ad inf.})$$

and let n be the number of observations required by the sequential test, i.e., n is the smallest integer for which $Z_n = z_1 + \cdots + z_n$ is either $\geq \log A$ or $\leq \log B$. To determine the expected value $E(n)$ of n under any hypothesis H we shall consider a fixed positive integer N. The sum $Z_N = z_1 + \cdots + z_N$ can be split in two parts as follows

$$Z_N = Z_n + Z'_n \tag{4.1}$$

where $Z'_n = z_{n+1} + \cdots + z_N$ if $n \leq N$ and $Z'_n = Z_N - Z_n$ if $n > N$. Taking expected values on both sides of (4.1) we obtain

$$NEz = EZ_n + EZ'_n. \tag{4.2}$$

Since the probability that $n > N$ converges to zero as $N \to \infty$, and since $| Z'_n | < 2(\log A + | \log B |)$ if $n > N$, it can be seen that

$$\lim_{N=\infty} [EZ'_n - E(N - n)Ez] = 0. \tag{4.3}$$

From (4.2) and (4.3) it follows that

$$EZ_n = EnEz. \tag{4.4}$$

Hence

$$En = \frac{EZ_n}{Ez} \tag{4.5}$$

Let E^*Z_n be the conditional expected value of Z_n under the restriction that the sequential analysis leads to the acceptance of H_0, i.e. that $Z_n \leq \log B$. Similarly, let $E^{**}Z_n$ be the conditional expected value of Z_n under the restriction that H_1 is accepted, i.e., that $Z_n \geq \log A$. Since γ is the probability that $Z_n \geq \log A$, we have

$$EZ_n = (1 - \gamma)E^*Z_n + \gamma E^{**}Z_n. \tag{4.6}$$

From (4.5) and (4.6) we obtain

$$En = \frac{(1-\gamma)E^{*}Z_n + \gamma E^{**}Z_n}{Ez}. \tag{4.7}$$

The exact value of EZ_n, and therefore also the exact value of En, can be computed if z can take only integral multiples of a constant d, since in this case the exact probability distribution of Z_n was obtained (see equation (3.47)). If z does not satisfy the above restriction, it is still possible to obtain arbitrarily fine approximations to the value of EZ_n, since the distribution of z can be approximated to any desired degree by a discrete distribution of the type mentioned above if the constant d is chosen sufficiently small.

If both $|Ez|$ and the standard deviation of z are small, $E^{*}Z_n$ is very nearly equal to $\log B$ and $E^{**}Z_n$ is very nearly equal to $\log A$. Hence in this case we can write

$$En \sim \frac{(1-\gamma)\log B + \gamma \log A}{Ez}. \tag{4.8}$$

To judge the goodness of the approximation given in (4.8) we shall derive lower and upper limits for En by deriving lower and upper limits for $E^{*}Z_n$ and $E^{**}Z_n$. Let r be a non-negative variable and let

$$\xi = \operatorname*{Max}_{r} E(z - r \mid z \geq r) \qquad (r \geq 0) \tag{4.9}$$

and

$$\xi' = \operatorname*{Min}_{r} E(z + r \mid z + r \leq 0). \qquad (r \geq 0) \tag{4.10}$$

It is easy to see that

$$\log A \leq E^{**}Z_n \leq \log A + \xi \tag{4.11}$$

and

$$\log B + \xi' \leq E^{*}Zn \leq \log B. \tag{4.12}$$

We obtain from (4.7), (4.11) and (4.12)

$$\frac{(1-\gamma)(\log B + \xi') + \gamma \log A}{Ez} \leq En \leq \frac{(1-\gamma)\log B + \gamma(\log A + \xi)}{Ez} \tag{4.13}$$

and

if $Ez > 0$

$$\frac{(1-\gamma)\log B + \gamma(\log A + \xi)}{Ez} \leq En \leq \frac{(1-\gamma)(\log B + \xi') + \gamma \log A}{Ez} \tag{4.14}$$

if $Ez < 0$.

4.2. *Calculation of the quantities ξ and ξ' for binomial and normal distributions.* Let X be a random variable which can take only the values 0 and 1. Let the probability that $X = 1$ be p_i if H_i is true ($i = 0, 1$), and p if H is true. Denote

$1 - p$ by q and $1 - p_i$ by q_i $(i = 0, 1)$. Then $f_i(1) = p_i$, $f_i(0) = q_i$, $f(1) = p$ and $f(0) = q$. It can be assumed without loss of generality that $p_1 > p_0$. It is clear that $\log \frac{f_1(x)}{f_0(x)} > 0$ implies that $x = 1$ and consequently $\log \frac{f_1(x)}{f_0(x)} = \log \frac{f_1(1)}{f_0(1)} = \log \frac{p_1}{p_0}$. Hence

$$\xi = \operatorname*{Max}_{r} E(z - r \mid z \geq r) = \log \frac{p_1}{p_0}. \tag{4.15}$$

Since $\log \frac{f_1(x)}{f_0(x)} \leq 0$ implies that $x = 0$, we have

$$\xi' = \operatorname*{Min}_{r} E(z + r \mid z + r \leq 0) = \log \frac{q_1}{q_0}. \tag{4.16}$$

Now we shall calculate the values ξ and ξ' if X is normally distributed. Let

$$f_i(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\theta_i)^2/2} \qquad (i = 0, 1)\ (\theta_1 > \theta_0)$$

and

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\theta)^2/2}.$$

We may assume without loss of generality that $\theta_0 = -\Delta$ and $\theta_1 = \Delta$ where $\Delta > 0$, since this can always be achieved by a translation. Then

$$z = \log \frac{f_1(x)}{f_0(x)} = 2\Delta x. \tag{4.17}$$

Denote $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2}$ by $\Phi(x)$ and $\frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{1}{2} t^2}\, dt$ by $G(x)$. Let $t = x - \theta$. Then $z = 2\Delta(t + \theta)$ and

$$\begin{aligned} E(z - r \mid z - r \geq 0) &= 2\Delta E\left(t + \theta - \frac{r}{2\Delta} \,\middle|\, t + \theta - \frac{r}{2\Delta} \geq 0\right) \\ &= \frac{2\Delta}{G(t_0)} \int_{t_0}^\infty (t - t_0)\Phi(t)\, dt = \frac{2\Delta}{G(t_0)} [-t_0 G(t_0) + \Phi(t_0)] \end{aligned} \tag{4.18}$$

where

$$t_0 = \frac{r}{2\Delta} - \theta. \tag{4.19}$$

In section 3.5 (see equation (3.70)) it was proved that $\frac{\Phi(t_0)}{G(t_0)} - t_0$ is a monotonically decreasing function of t_0. Hence the maximum of $E(z - r \mid z - r \geq 0)$ is reached for $r = 0$ and consequently

$$\xi = \frac{2\Delta}{G(-\theta)} [\theta G(-\theta) + \Phi(-\theta)] = 2\Delta\left[\theta + \frac{\Phi(-\theta)}{G(-\theta)}\right]. \tag{4.20}$$

Now we shall calculate ξ'. We have

$$\begin{aligned}\xi' &= \operatorname*{Min}_r E(z + r \mid z + r \leq 0) = -\operatorname*{Max}_r E(-z - r \mid -z - r \geq 0) \\ &= -2\Delta \operatorname*{Max}_r E\left(-x - \frac{r}{2\Delta} \,\middle|\, -x - \frac{r}{2\Delta} \geq 0\right).\end{aligned} \tag{4.21}$$

Let $t = -x + \theta$ and $t_0 = \dfrac{r}{2\Delta} + \theta$. Then

$$\begin{aligned}E\left(-x - \frac{r}{2\Delta} \,\middle|\, -x - \frac{r}{2\Delta} \geq 0\right) &= E(t - t_0 \mid t - t_0 \geq 0) \\ &= \frac{1}{G(t_0)} \int_{t_0}^{\infty} (t - t_0)\Phi(t)\, dt = \frac{\Phi(t_0)}{G(t_0)} - t_0 .\end{aligned} \tag{4.22}$$

Since this is a monotonically decreasing function of t_0, we have

$$\operatorname*{Max}_r E\left(-x - \frac{r}{2\Delta} \,\middle|\, -x - \frac{r}{2\Delta} \geq 0\right) = \frac{\Phi(\theta)}{G(\theta)} - \theta. \tag{4.23}$$

From (4.21) and (4.23) we obtain

$$\xi' = -2\Delta\left[\frac{\Phi(\theta)}{G(\theta)} - \theta\right]. \tag{4.24}$$

4.3. *Saving in the number of observations as compared with the current test procedure.* We consider the case of a normally distributed variate, such that

$$f_0(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(x-\theta_0)^2}$$

and

$$f_1(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(x-\theta_1)^2} \qquad (\theta_1 \neq \theta_0).$$

Denote by $n(\alpha, \beta)$ the minimum number of observations necessary in the current most powerful test for the probabilities of errors of the first and second kinds to be α and β, respectively, or less.

We shall calculate the number of observations required by the most powerful test. It can be assumed without loss of generality that $\theta_0 \leq \theta_1$ According to the current most powerful test procedure the hypothesis H_0 is accepted if $\bar{x} \leq d$ and the hypothesis H_1 is accepted if $\bar{x} > d$, where $\bar{x}$ is the arithmetic mean of the observations and d is a properly chosen constant. The probability of an error of the first kind is given by $G[\sqrt{n}(d - \theta_0)]$ and the probability of an error of the second kind is given by $1 - G[\sqrt{n}(d - \theta_1)]$ where $G(t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_t^{\infty} e^{-x^2/2}\, dx$. To equate these probabilities to α and β, respectively, the quantities d and n must satisfy

$$G[\sqrt{n}(d - \theta_0)] = \alpha \tag{4.25}$$

and

$$1 - G[\sqrt{n}(d - \theta_1)] = \beta. \tag{4.26}$$

Denote by λ_0 and λ_1 the values for which $G(\lambda_0) = \alpha$ and $G(\lambda_1) = 1 - \beta$. Then we have

$$\sqrt{n}(d - \theta_0) = \lambda_0 \tag{4.27}$$

and

$$\sqrt{n}(d - \theta_1) = \lambda_1 . \tag{4.28}$$

Subtracting (4.27) from (4.28) we obtain

$$\sqrt{n}(\theta_0 - \theta_1) = \lambda_1 - \lambda_0 . \tag{4.29}$$

From (4.29)

$$n = n(\alpha, \beta) = \frac{(\lambda_1 - \lambda_0)^2}{(\theta_0 - \theta_1)^2}. \tag{4.30}$$

If the expression on the right hand side of (4.30) is not an integer, $n(\alpha, \beta)$ is the smallest integer in excess.

In the sequential probability ratio test we put $A = a(\alpha, \beta) = \frac{1 - \beta}{\alpha}$ and $B = b(\alpha, \beta) = \frac{\beta}{1 - \alpha}$. Then the probability of an error of the first (second) kind cannot exceed $\alpha(\beta)$ except by a negligible amount. Let $A(\alpha, \beta)$ and $B(\alpha, \beta)$ be the values of A and B for which the probabilities of errors of the first and second kinds become exactly equal to α and β, respectively. It has been shown in Section 3.2 that $A(\alpha, \beta) \leq a(\alpha, \beta)$ and $B(\alpha, \beta) \geq b(\alpha, \beta)$. Thus, the expected values $E_1(n)$ and $E_0(n)$ are only increased by putting $A = a\ (\alpha, \beta)$ and $B = b\ (\alpha, \beta)$ instead of $A = A\ (\alpha, \beta)$ and $B = B\ (\alpha, \beta)$.

Consider the case where $| \theta_1 - \theta_0 |$ is small so that the quantities ξ and ξ' can be neglected. Thus, we shall use the approximation (4.8). Since $\gamma = \alpha$ if $H = H_0$ and $\gamma = 1 - \beta$ if $H = H_1$, we obtain from (4.8)

$$E_1(n) = \frac{a^*}{E_1(z)} - \beta \frac{a^* + |b^*|}{E_1(z)} \tag{4.31}$$

and

$$E_0(n) = \frac{-b^*}{E_0(-z)} - \alpha \frac{-b^* + a^*}{E_0(-z)} \tag{4.32}$$

where $a^* = \log a(\alpha, \beta) = \log \frac{1 - \beta}{\alpha}$ and $b^* = \log b(\alpha, \beta) = \log \frac{\beta}{1 - \alpha}$. Since

$$E_1(z) = \tfrac{1}{2}(\theta_0 - \theta_1)^2 \tag{4.33}$$

and

(4.34) $$E_0(-z) = \tfrac{1}{2}(\theta_0 - \theta_1)^2,$$

it follows from (4.30), (4.31) and (4.32) that $\dfrac{E_1(n)}{n(\alpha, \beta)}$ and $\dfrac{E_0(n)}{n(\alpha, \beta)}$ are independent of the parameters θ_0 and θ_1.

TABLE 1

Average percentage saving of sequential analysis, as compared with current most powerful test for testing mean of a normally distributed variate

A. When alternative hypothesis is true:

β \ α	.01	.02	.03	04	.05
.01	58	60	61	62	63
.02	54	56	57	58	59
.03	51	53	54	55	55
.04	49	50	51	52	53
.05	47	49	50	50	51

B. When null hypothesis is true:

β \ α	.01	.02	.03	04	.05
.01	58	54	51	49	47
.02	60	56	53	50	49
.03	61	57	54	51	50
.04	62	58	55	52	50
.05	63	59	55	53	51

The average saving of the sequential analysis as compared with the current method is $100\left(1 - \dfrac{E_1(n)}{n(\alpha, \beta)}\right)$ per cent if H_1 is true, and $100\left(1 - \dfrac{E_0(n)}{n(\alpha, \beta)}\right)$ per cent if H_0 is true. In Table 1 the expression $100\left(1 - \dfrac{E_1(n)}{n(\alpha, \beta)}\right)$ is shown in Panel A, and the expression $100\left(1 - \dfrac{E_0(n)}{n(\alpha, \beta)}\right)$ in Panel B, for several values of α and β. Because of the symmetry of the normal distribution, Panel B is obtained from Panel A simply by interchanging α and β.

As can be seen from the table, for the range of α and β from .01 to .05 (the range most frequently employed), the sequential process leads to an average saving of at least 47 per cent in the necessary number of observations as compared with the current procedure. The true saving is slightly greater than shown in the table, since $E_i(n)$ calculated under the condition that $A = a\ (\alpha, \beta)$ and $B = b\ (\alpha, \beta)$ is greater than $E_i(n)$ calculated under the condition that $A = A\ (\alpha, \beta)$ and $B = B\ (\alpha, \beta)$.

4.4. *The characteristic function, the moments and the distribution of the number of observations necessary for reaching a decision.* It was shown in [4] (see equation (15) in [4]) that the following fundamental identity holds

$$E\{e^{Z_n t}[\varphi(t)]^{-n}\} = 1 \qquad (\varphi(t) = Ee^{zt}) \tag{4.35}$$

for all points t of the complex plane for which $\varphi(t)$ exists and $|\varphi(t)| \geq 1$. The symbol n denotes the number of observations required by the sequential test, i.e., n is the smallest positive integer for which Z_n is either $\geq \log A$ or $\leq \log B$, and $\varphi(t)$ denotes the moment generating function of z.

On the basis of the identity (4.35) the exact characteristic function of n is derived in section 7 of [4] in the case when z can take only integral multiples of a constant. If the number of different values which Z_n can take is large, the calculation of the exact characteristic function is cumbersome, because a large number of simultaneous linear equations have to be solved. However, if $|Ez|$ and σ_z are small so that $|Z_n - \log A|$ (when $Z_n \geq \log A$) and $|Z_n - \log B|$ (when $Z_n \leq \log B$) can be neglected, the calculation of the characteristic function is much simpler, as was shown in [4]. We shall briefly state the results obtained in [4]. Let h be the real value $\neq 0$ for which $\varphi(h) = 1$. Furthermore let $t = t_1(\tau)$ and $t = t_2(\tau)$ be the roots of the equation in t

$$-\log \varphi(t) = \tau$$

such that $\lim_{\tau=0} t_1(\tau) = 0$ and $\lim_{\tau=0} t_2(\tau) = h$. Finally, let $\psi_1(\tau)$ the characteristic function of the conditional distribution of n under the restriction that $Z_n \geq \log A$, and $\psi_2(\tau)$ the characteristic function of the conditional distribution of n under the restriction that $Z_n \leq \log B$. Then, if $|Z_n - \log A|$ (when $Z_n \geq \log A$) and $|Z_n - \log B|$ (when $Z_n \leq \log B$) can be neglected, $\psi_1(\tau)$ and $\psi_2(\tau)$ are the solutions of the linear equations

$$\gamma\psi_1(\tau)A^{t_1(\tau)} + (1 - \gamma)\psi_2(\tau)B^{t_1(\tau)} = 1 \tag{4.36}$$

and

$$\gamma\psi_1(\tau)A^{t_2(\tau)} + (1 - \gamma)\psi_2(\tau)B^{t_2(\tau)} = 1 \tag{4.37}$$

where

$$\gamma = P(Z_n \geq \log A) = \frac{1 - B^h}{A^h - B^h}.$$

The characteristic function of the unconditional distribution of n is

$$\psi(\tau) = \gamma\psi_1(\tau) + (1 - \gamma)\psi_2(\tau). \tag{4.38}$$

As an illustration we shall determine $\psi_1(\tau)$, $\psi_2(\tau)$ and $\psi(\tau)$ when z has a normal distribution. Then we have

$$- \log \varphi(t) = -(Ez)t - \frac{\sigma_z^2}{2} t^2 = \tau.$$

Hence

$$h = -\frac{2Ez}{\sigma_z^2} \tag{4.39}$$

$$\begin{aligned} t_1(\tau) &= \frac{1}{\sigma_z^2}\left(-Ez + \sqrt{(Ez)^2 - 2\sigma_z^2 \tau}\right). \\ t_2(\tau) &= \frac{1}{\sigma_z^2}\left(-Ez - \sqrt{(Ez)^2 - 2\sigma_z^2 \tau}\right). \end{aligned} \tag{4.40}$$

From (4.36), (4.37) and (4.38) we obtain

$$\gamma\psi_1(\tau) = \frac{B^{g_2} - B^{g_1}}{A^{g_1}B^{g_2} - A^{g_2}B^{g_1}}, \tag{4.41}$$

$$(1 - \gamma)\psi_2(\tau) = \frac{A^{g_1} - A^{g_2}}{A^{g_1}B^{g_2} - A^{g_2}B^{g_1}} \tag{4.42}$$

and

$$\psi(\tau) = \frac{A^{g_1} + B^{g_2} - A^{g_2} - B^{g_1}}{A^{g_1}B^{g_2} - A^{g_2}B^{g_1}} \tag{4.43}$$

where

$$g_1 = \frac{1}{\sigma_z^2}\left(-Ez + \sqrt{(Ez)^2 - 2\sigma_z^2 \tau}\right) \tag{4.44}$$

and

$$g_2 = \frac{1}{\sigma_z^2}\left(-Ez - \sqrt{(Ez)^2 - 2\sigma_z^2 \tau}\right). \tag{4.45}$$

For any positive integer r the r-th moment of n i.e., $E(n^r)$ is equal to the r-th derivative of $\psi(\tau)$ taken at $\tau = 0$. Let $E^*(n^r)$ be the conditional expected value of n^r under the restriction that $Z_n \leq \log B$, and let $E^{**}(n^r)$ be the conditional expected value of n^r under the restriction that $Z_n \geq \log A$. Then

$$E^*(n^r) = \left.\frac{d^r \psi_2(\tau)}{d\tau^r}\right|_{\tau=0} \quad \text{and} \quad E^{**}(n^r) = \left.\frac{d^r \psi_1(\tau)}{d\tau^r}\right|_{\tau=0}. \tag{4.46}$$

It may be of interest to note that $\left.\frac{d^r\psi_k(\tau)}{d\tau^r}\right|_{\tau=0}$ $(k = 1, 2)$ and therefore also the moments of n can be obtained from the identity (4.35) directly by successive differentiation. In fact, the identity (4.35) can be written as (neglecting the excess of Z_n over the boundaries $\log A$ and $\log B$)

$$\gamma A^t\psi_1[-\log\varphi(t)] + (1 - \gamma)B^t\psi_2[-\log\varphi(t)] = 1. \tag{4.47}$$

Taking the first r derivatives of (4.47) with respect to t at $t = 0$ and $t = h$ we obtain a system of $2r$ linear equations in the $2r$ unknowns $\left.\frac{d^j\psi_k(\tau)}{d\tau^j}\right|_{\tau=0}$ ($k = 1, 2; j = 1, \cdots, r$) from which these unknowns can be determined. For example, $\left.\frac{d\psi_k(\tau)}{d\tau}\right|_{\tau=0}$ ($k = 1, 2$) can be determined as follows: Taking the first derivative of (4.47) with respect to t and denoting $\frac{d^r\psi_k(\tau)}{d\tau^r}$ by $\psi_k^{(r)}(\tau)$ we obtain

$$\begin{aligned}\gamma(\log A)A^t\psi_1[-\log\varphi(t)] - \gamma A^t\frac{\varphi'(t)}{\varphi(t)}\psi_1^{(1)}[-\log\varphi(t)]&\\ + (1-\gamma)(\log B)B^t\psi_2[-\log\varphi(t)]&\\ - (1-\gamma)B^t\frac{\varphi'(t)}{\varphi(t)}\psi_2^{(1)}[-\log\varphi(t)] &= 0.\end{aligned}\tag{4.48}$$

Putting $t = 0$ and $t = h$ we obtain the equations

$$\gamma\log A - \gamma\frac{\varphi'(0)}{\varphi(0)}\psi_1^{(1)}(0) + (1-\gamma)\log B - (1-\gamma)\frac{\varphi'(0)}{\varphi(0)}\psi_2^{(1)}(0) = 0\tag{4.49}$$

and

$$\begin{aligned}\gamma(\log A)A^h - \gamma A^h\frac{\varphi'(h)}{\varphi(h)}\psi_1^{(1)}(0)&\\ + (1-\gamma)(\log B)B^h - (1-\gamma)B^h\frac{\varphi'(h)}{\varphi(h)}\psi_2^{(1)}(0) &= 0\end{aligned}\tag{4.50}$$

from which $\psi_1^{(1)}(0)$ and $\psi_2^{(1)}(0)$ can be determined.

The distribution of n can be obtained by inverting the characteristic function of $\psi(\tau)$. This was done in [4] (neglecting the excess of Z_n over $\log A$ and $\log B$) in the case when z is normally distributed. The results obtained in [4] can be briefly stated as follows: If $B = 0$, or if $B > 0$ and $A = \infty$, the distribution of n is a simple elementary function. If $B = 0$ and $Ez > 0$, the distribution of $m = \frac{1}{2\sigma_z^2}(Ez)^2 n$ is given by

$$F(m)\,dm = \frac{c}{2\Gamma(\frac{1}{2})m^{\frac{3}{2}}}e^{-c^2/4m-m+c}\,dm \qquad (0 \le m < \infty)\tag{4.51}$$

where

$$c = \frac{1}{\sigma_z^2}(Ez)\log A.\tag{4.52}$$

If $B > 0$, $A = \infty$ and $Ez < 0$ the distribution of $m = \frac{1}{2\sigma_z^2}(Ez)^2 n$ is given by the expression we obtain from (4.51) if we substitute $\frac{1}{\sigma_z^2}(Ez)\log B$ for c.

If $B > 0$ and $A < \infty$, the distribution of m is given by an infinite series where each term is of the form (4.51) (see equation (76) in [4]).

Since m is a discrete variable, it may seem paradoxical that we obtained a probability density function for m. However, the explanation lies in the fact that we neglected the excess of Z_n over $\log A$ and $\log B$ which is zero only in the limiting case when Ez and σ_z approach zero.

The distribution of m given in (4.51) can be used as a good approximation to the exact distribution of m even if $B > 0$, provided that the probability that $Z_n \geq \log A$ is nearly equal to 1.

It was pointed out in [4] that if $| Ez |$ and σ_z are sufficiently small, the distribution of n determined under the assumption that z is normally distributed will be a good approximation to the exact distribution of n even if z is not normally distributed.

4.5. *Lower limit of the probability that the sequential process will terminate with a number of trials less than or equal to a given number.* Let $P_i(n_0)$ be the probability that the sequential process will terminate at a value $n \leq n_0$, calculated under H_i $(i = 0, 1)$. Let

$$\bar{P}_0(n_0) = P_0\left[\sum_{\alpha=1}^{n_0} z_\alpha \leq \log B\right] \tag{4.53}$$

and

$$\bar{P}_1(n_0) = P_1\left[\sum_{\alpha=1}^{n_0} z_\alpha \geq \log A\right]. \tag{4.54}$$

It is clear that

$$\bar{P}_i(n_0) \leq P_i(n_0) \qquad (i = 0, 1). \tag{4.55}$$

For calculating $\bar{P}_i(n_0)$ we shall assume that n_0 is sufficiently large so that $\sum_{\alpha=1}^{n_0} z_\alpha$ can be regarded as normally distributed. Let $G(\lambda)$ be defined by

$$G(\lambda) = \frac{1}{\sqrt{2\pi}} \int_\lambda^\infty e^{-\frac{1}{2}t^2}\, dt. \tag{4.56}$$

Furthermore, let

$$\lambda_1(n_0) = \frac{\log A - n_0 E_1(z)}{\sqrt{n_0}\, \sigma_1(z)} \tag{4.57}$$

and

$$\lambda_0(n_0) = \frac{\log B - n_0 E_0(z)}{\sqrt{n_0}\, \sigma_0(z)} \tag{4.58}$$

where $\sigma_i(z)$ is the standard deviation of z under H_i. Then

$$\bar{P}_1(n_0) = G[\lambda_1(n_0)] \tag{4.59}$$

and

$$\bar{P}_0(n_0) = 1 - G[\lambda_0(n_0)]. \tag{4.60}$$

Hence we have the inequalities

$$P_1(n_0) \geq G[\lambda_1(n_0)] \tag{4.61}$$

and

$$P_0(n_0) \geq 1 - G[\lambda_0(n_0)]. \tag{4.62}$$

Putting $\log A = \log \frac{1-\beta}{\alpha}$ and $\log B = \log \frac{\beta}{1-\alpha}$, Table 2 shows the values of $\bar{P}_1(n_0)$ and $\bar{P}_0(n_0)$ corresponding to different pairs (α, β) and different values of n_0. In these calculations it has been assumed that the distribution under H_0 is a normal distribution with mean zero and unit variance, and the distribution under H_1 is a normal distribution with mean θ and unit variance. For each pair (α, β) the value of θ was determined so that the number of observations required by the current most powerful test of strength (α, β) is equal to 1000.

TABLE 2

Lower bound of the probability that a sequential analysis will terminate within various numbers of trials, when the most powerful current test requires exactly 1000 trials*

Number of trials	$\alpha = .01$ and $\beta = .01$		$\alpha = .01$ and $\beta = .05$		$\alpha = .05$ and $\beta = .05$	
	Alternative hypothesis true	Null hypothesis true	Alternative hypothesis true	Null hypothesis true	Alternative hypothesis true	Null hypothesis true
1000	.910	.910	.799	.891	.773	.773
1200	.950	.950	.871	.932	.837	.837
1400	.972	.972	.916	.957	.883	.883
1600	.985	.985	.946	.972	.915	.915
1800	.991	.991	.965	.982	.938	.938
2000	.995	.995	977	.989	.955	.955
2200	.997	.997	.985	.993	.967	.967
2400	.999	.999	.990	.995	.976	976
2600	.999	.999	.994	.997	.982	.982
2800	1.00	1.00	.996	.998	.987	.987
3000	1.00	1.00	.997	.999	.990	.990

* The probabilities given are lower bounds for the true probabilities. They relate to a test of the mean of a normally distributed variate, the difference between the null and alternative hypothesis being adjusted for each pair of values of α and β so that the number of trials required under the most powerful current test is exactly 1000.

4.6. *Truncated-sequential analysis.* In some applications a definite upper bound for the number of observations may be desirable. Thus, a certain integer n_0 is chosen so that if the sequential process does not lead to a final decision for $n \leq n_0$, a new rule is given for the acceptance or rejection of H_0 at the stage $n = n_0$.

A simple and reasonable rule for the acceptance or rejection of H_0 at the stage $n = n_0$ can be given as follows: If $\sum_{\alpha=1}^{n_0} z_\alpha \leq 0$ we accept H_0 and if $\sum_{\alpha=1}^{n_0} z_\alpha > 0$

we accept H_1. By thus truncating the sequential process we change, however, the probabilities of errors of the first and second kinds. Let α and β be the probabilities of errors of the first and second kinds, respectively, if the sequential test is not truncated. Let $\alpha(n_0)$ and $\beta(n_0)$ be the probabilities of errors of the first and second kinds if the test is truncated at $n = n_0$. We shall derive upper bounds for $\alpha(n_0)$ and $\beta(n_0)$.

First we shall derive an upper bound for $\alpha(n_0)$. Let $\rho_0(n_0)$ be the probability (under the null hypothesis) that the following three conditions are simultaneously fulfilled:

(i) $$\log B < \sum_{\alpha=1}^{n} z_\alpha < \log A \qquad \text{for } n = 1, \cdots, n_0 - 1$$

(ii) $$0 < \sum_{\alpha=1}^{n_0} z_\alpha < \log A$$

(iii) continuing the sequential process beyond n_0, it terminates with the acceptance of H_0.

It is clear that

$$\alpha(n_0) \leq \alpha + \rho_0(n_0). \tag{4.63}$$

Let $\bar{\rho}_0(n_0)$ be the probability (under the null hypothesis) that $0 < \sum_{\alpha=1}^{n_0} z_\alpha < \log A$. Then obviously

$$\rho_0(n_0) \leq \bar{\rho}_0(n_0)$$

and consequently

$$\alpha(n_0) \leq \alpha + \bar{\rho}_0(n_0). \tag{4.64}$$

Let $\rho_1(n_0)$ be the probability under the alternative hypothesis that the following three conditions are simultaneously fulfilled:

(i) $$\log B < \sum_{\alpha=1}^{n} z_\alpha < \log A \qquad \text{for } n = 1, \cdots, n_0 - 1$$

(ii) $$\log B < \sum_{\alpha=1}^{n_0} z_\alpha \leq 0$$

(iii) continuing the sequential process beyond n_0, it terminates with the acceptance of H_1.

It is clear that

$$\beta(n_0) \leq \beta + \rho_1(n_0) \tag{4.65}$$

Let $\bar{\rho}_1(n_0)$ be the probability (under the alternative hypothesis) that $\log B < \sum_{\alpha=1}^{n_0} z_\alpha \leq 0$. Then $\rho_1(n_0) \leq \bar{\rho}_1(n_0)$ and consequently

$$\beta(n_0) \leq \beta + \bar{\rho}_1(n_0). \tag{4.66}$$

Let

$$\nu_1 = \frac{-n_0 E_0(z)}{\sqrt{n_0}\,\sigma_0(z)}$$

$$\nu_2 = \frac{\log A - n_0 E_0(z)}{\sqrt{n_0}\,\sigma_0(z)}, \qquad \nu_3 = \frac{-n_0 E_1(z)}{\sqrt{n_0}\,\sigma_1(z)}, \qquad \nu_4 = \frac{\log B - n_0 E_1(z)}{\sqrt{n_0}\,\sigma_1(z)}$$

where $\sigma_i(z)$ is the standard deviation of z under H_i $(i = 0, 1)$. Then

$$\bar{\rho}_0(n_0) = G(\nu_1) - G(\nu_2) \tag{4.67}$$

and

$$\bar{\rho}_1(n_0) = G(\nu_4) - G(\nu_3). \tag{4.68}$$

From (4.64), (4.66), (4.67) and (4.68) we obtain

$$\alpha(n_0) \leq \alpha + G(\nu_1) - G(\nu_2) \tag{4.69}$$

and

$$\beta(n_0) \leq \beta + G(\nu_4) - G(\nu_3). \tag{4.70}$$

The upper bounds given in (4.69) and (4.70) may considerably exceed $\alpha(n_0)$ and $\beta(n_0)$, respectively. It would be desirable to find closer limits.

Table 3 shows the values of the upper bounds of $\alpha(n_0)$ and $\beta(n_0)$ given by formulas (4.69) and (4.70) corresponding to different pairs (α, β) and different values of n_0. In these calculations we have put $\log A = \log \frac{1-\beta}{\alpha}$, $\log B = \log \frac{\beta}{1-\alpha}$ and assumed that the distribution under H_0 is a normal distribution with mean zero and unit variance, and the distribution under H_1 is a normal distribution with mean θ and unit variance. For each pair (α, β) the value of θ has been determined so that the number of observations required by the current most powerful test of strength (α, β) is equal to 1000.

It seems to the author that the upper limits given in (4.69) and (4.70) are considerably above the true $\alpha(n_0)$ and $\beta(n_0)$ respectively, when n_0 is not much higher than the value of n needed for the current most powerful test.

4.7 *Efficiency of the sequential probability ratio test.* Let S be any sequential test for which the probability of an error of the first kind is α, the probability of an error of the second kind is β and the probability that the test procedure will eventually terminate is one. Let S' be the sequential probability ratio test whose strength is equal to that of S. We shall prove that the sequential probability ratio test is an optimum test, i.e., that $E_i(n \mid S) \geq E_i(n \mid S')$ $(i = 0, 1)$, if for S' the excess of Z_n over $\log A$ and $\log B$ can be neglected. This excess is exactly zero if z can take only the values d and $-d$ and if $\log A$ and $\log B$ are integral multiples of d. In any other case the excess will not be identically zero. However, if $\mid Ez \mid$ and σ_z are sufficiently small, the excess of Z_n over $\log A$ and $\log B$ is negligible.

For any random variable u we shall denote by $E_i^*(u \mid S)$ the conditional expected value of u under the hypothesis H_i $(i = 0, 1)$ and under the restriction

that H_0 is accepted. Similarly, let $E_i^{**}(u \mid S)$ be the conditional expected value of u under the hypothesis H_i ($i = 0, 1$) and under the restriction that H_1 is accepted. In the notations for these expected values the symbol S stands for

TABLE 3

Effect on risks of error of truncating a sequential analysis at a predetermined number of trials*

Number of trials	$\alpha = .01$ and $\beta = .01$		$\alpha = .01$ and $\beta = .05$		$\alpha = .05$ and $\beta = .05$	
	Upper bound of effective α	Upper bound of effective β	Upper bound of effective α	Upper bound of effective β	Upper bound of effective α	Upper bound of effective β
1000	.020	.020	.033	.070	.095	.095
1200	.015	.015	.024	.063	.082	.082
1400	.013	.013	.019	.058	.072	.072
1600	.012	.012	.016	.055	.066	.066
1800	.011	.011	.014	.053	.062	.062
2000	.010	.010	.012	.052	.058	.058
2200	.010	.010	.012	.051	.056	.056
2400	.010	.010	.011	.051	.055	.055
2600	.010	.010	.011	.051	.053	.053
2800	.010	.010	.010	.050	.053	.053
3000	.010	.010	.010	.050	.052	.052

* If the sequential analysis is based on the values α and β shown, but a decision is made at n_0 trials even when the normal sequential criteria would require a continuation of the process, the realized values of α and β will not exceed the tabular entries. The table relates to a test of the mean of a normally distributed variate, the difference between the null and alternative hypotheses being adjusted for each pair (α, β) so that the number of trials required by the current test is 1000.

the sequential test used. Denote by $Q_i(S)$ the totality of all samples for which the test S leads to the acceptance of H_i. Then we have

$$E_0^*\left(\frac{p_{1n}}{p_{0n}} \mid S\right) = \frac{P_1[Q_0(S)]}{P_0[Q_0(S)]} = \frac{\beta}{1-\alpha} \tag{4.71}$$

$$E_0^{**}\left(\frac{p_{1n}}{p_{0n}} \mid S\right) = \frac{P_1[Q_1(S)]}{P_0[Q_1(S)]} = \frac{1-\beta}{\alpha} \tag{4.72}$$

$$E_1^*\left(\frac{p_{0n}}{p_{1n}} \mid S\right) = \frac{P_0[Q_0(S)]}{P_1[Q_0(S)]} = \frac{1-\alpha}{\beta} \tag{4.73}$$

and

$$E_1^{**}\left(\frac{p_{0n}}{p_{1n}} \mid S\right) = \frac{P_0[Q_1(S)]}{P_1[Q_1(S)]} = \frac{\alpha}{1-\beta}. \tag{4.74}$$

To prove the efficiency of the sequential probability ratio test, we shall first derive two lemmas.

LEMMA 1. *For any random variable u the inequality*

$$e^{Eu} \leq Ee^{u} \tag{4.75}$$

holds.

PROOF: Inequality (4.75) can be written as

$$1 \leq Ee^{u'} \tag{4.76}$$

where $u' = u - Eu$. Lemma 1 is proved if we show that (4.76) holds for any random variable u' with zero mean. Expanding $e^{u'}$ in a Taylor series around $u' = 0$, we obtain

$$e^{u'} = 1 + u' + \tfrac{1}{2}u'^2e^{\xi(u')} \quad \text{where } 0 \leq \xi(u') \leq u'. \tag{4.77}$$

Hence

$$Ee^{u'} = 1 + \tfrac{1}{2}E[u'^2e^{\xi(u')}] \geq 1 \tag{4.78}$$

and Lemma 1 is proved.

LEMMA 2. *Let S be a sequential test such that there exists a finite integer N with the property that the number n of observations required for the test is $\leq N$. Then*

$$E_i(n \mid S) = \frac{E_i\left(\log \frac{p_{1n}}{p_{0n}}\middle| S\right)}{E_i(z)} \qquad (i = 0, 1). \tag{4.79}$$

The proof is omitted, since it is essentially the same as that of equation (4.5) for the sequential probability ratio test.

On the basis of Lemmas 1 and 2 we shall be able to derive the following theorem.

THEOREM. *Let S be any sequential test for which the probability of an error of the first kind is α, the probability of an error of the second kind is β and the probability that the test procedure will eventually terminate is equal to one. Then*

$$E_0(n \mid S) \geq \frac{1}{E_0(z)}\left[(1 - \alpha)\log\frac{\beta}{1 - \alpha} + \alpha\log\frac{1 - \beta}{\alpha}\right] \tag{4.80}$$

and

$$E_1(n \mid S) \geq \frac{1}{E_1(z)}\left[\beta\log\frac{\beta}{1 - \alpha} + (1 - \beta)\log\frac{1 - \beta}{\alpha}\right] \tag{4.81}$$

PROOF: First we shall prove the theorem in the case when there exists a finite integer N such that n never exceeds N. According to Lemma 2 we have

$$\begin{aligned} E_0(n \mid S) &= \frac{1}{E_0(z)}\, E_0\left(\log\frac{p_{1n}}{p_{0n}}\middle| S\right) \\ &= \frac{1}{E_0(z)}\left[(1 - \alpha)E_0^*\left(\log\frac{p_{1n}}{p_{0n}}\middle| S\right) + \alpha E_0^{**}\left(\log\frac{p_{1n}}{p_{0n}}\middle| S\right)\right] \end{aligned} \tag{4.82}$$

PART II. SEQUENTIAL TEST OF A SIMPLE OR COMPOSITE HYPOTHESIS AGAINST A SET OF ALTERNATIVES

In Part I we have dealt with the problem of testing a simple hypothesis H_0 against a single alternative H_1. Here we shall consider the problem of testing a simple or composite hypothesis against a set of infinitely many alternatives. By a simple hypothesis we mean a hypothesis which specifies uniquely the probability distribution of the random variable x under consideration. A hypothesis is called composite, if it is not simple.

5. Test of a Simple Hypothesis Against One-sided Alternatives

5.1. *General remarks.* Let $f(x, \theta)$ be the probability density function of a random variable X, where θ is an unknown parameter. Suppose that it is required to test the simple hypothesis that $\theta = \theta_0$ and that the alternative values of θ are restricted to values $\theta > \theta_0$. Assume that it is desired to have a sequential test such that the probability of an error of the first kind is equal to a given α.

The probability of an error of the second kind is no longer a single value, but is a function of the true value of θ. If $f(x, \theta)$ is a continuous function of x and θ, the probability of an error of the second kind will be arbitrarily near $1 - \alpha$ if the true value of θ is sufficiently near θ_0. Hence, if α is small, the probability of an error of the second kind is necessarily large when the true value of θ is very near θ_0. In most practical applications we do not care if the probability of an error of the second kind is high when the true value of θ is very near θ_0, since in this case the error committed by accepting θ_0 is usually of very little importance. However, there will be a value $\theta_1 > \theta_0$ such that we wish the probability of an error of the second kind to be less than or equal to a given small positive value β whenever the true value of θ is greater than or equal to θ_1.

In this case we can proceed as follows: Consider the single alternative hypothesis H_1 that $\theta = \theta_1$. Construct a sequential test for testing $\theta = \theta_0$ against the single alternative H_1 such that the probability of an error of the first kind is α and the probability of an error of the second kind, i.e., the probability of accepting θ_0 when θ_1 is true, is β. If this sequential test has the further property that the probability of an error of the second kind is less than or equal to β whenever the true value of θ is greater than θ_1, then this sequential test provides a satisfactory solution of the problem of testing the hypothesis that $\theta = \theta_0$ against the set of alternatives $\theta > \theta_0$.

In most of the important cases occurring in practice, such as when X has a normal, binomial, or Poisson distribution, etc., the sequential probability ratio test for testing the hypothesis that $\theta = \theta_0$ against a single alternative θ_1 ($\theta_1 > \theta_0$) satisfies the condition that the probability of an error of the second kind is a monotonically decreasing function of θ in the domain $\theta > \theta_0$. Thus, in all these cases the sequential probability ratio test for testing the hypothesis that $\theta = \theta_0$ against a properly chosen alternative θ_1 provides a satisfactory solution of our problem.

The case in which the alternative values of θ are restricted to values less than θ_0 is entirely analogous to that in which the alternatives are restricted to values greater than θ_0, and need not be discussed separately.

It should be pointed out that the test procedure for testing $\theta = \theta_0$ against alternatives $\theta > \theta_0$, as described in this section, is also suitable for testing the composite hypothesis that $\theta \leq \theta_0$, provided that the probability of rejecting the null hypothesis is $\leq \alpha$ whenever the true value of θ is $\leq \theta_0$. This condition is fulfilled, for instance, when X has a normal, binomial or Poisson distribution.

5.2. *Application to binomial distributions.* 5.2.1. *Statement of the problem.* The case of a binomial distribution arises when the result of a single observation is a classification into one of two categories. For example, this is the situation in acceptance inspection of manufactured products, if each unit inspected is classified into one of the two categories, non-defective and defective. Let p denote the probability that an item belongs to a given category. The value of p is usually unknown. We shall deal here with the problem of testing the hypothesis that p does not exceed a given value p' against the alternative possibility that $p > p'$.

Since acceptance inspection of manufactured products is perhaps the most important and widest field of application of such a test procedure, we shall, in continuing the discussion, use the terminology of acceptance inspection. This, of course, does not mean that the test procedure is not applicable to other cases. Suppose that a lot containing a large number of units is submitted for sampling inspection. Let p denote the proportion of defective units contained in the lot. The probability that a unit drawn at random from the lot will be defective is equal to p. If m units are drawn at random from the lot, the probability that there will be d defectives among them is given by[13]

$$\frac{m!}{d!(m-d)!}\,p^d(1-p)^{m-d} \qquad (d = 0, 1, \cdots, m). \tag{5.1}$$

The probability distribution as given in (5.1) is called a binomial distribution.

The purpose of sampling inspection is to decide whether the lot should be accepted or rejected. It is clear that for high values of p we want to reject the lot and for low values of p we want to accept the lot. Thus, it will be possible to specify a particular value of p, say p', so that if $p \leq p'$ we wish to accept the lot, and if $p > p'$ we wish to reject the lot. Thus, our problem is to devise a proper sampling inspection plan for testing the hypothesis that $p \leq p'$.

5.2.2. *Tolerated risks for making a wrong decision.* No sampling inspection plan can guarantee that the correct decision will always be made, i.e., that the lot will always be accepted when $p \leq p'$ and the lot will always be rejected when $p > p'$, unless the lot is inspected completely. A complete inspection is usually

[13] Formula (5.1) is exact only if the lot contains infinitely many units. While the lot is always finite in practice, we shall assume that m is small as compared with the lot size so that formula (5.1) can be used.

rather uneconomical and one is willing to take some risk of making a wrong decision if this permits a reduction in the amount of inspection. Hence, recommendations as to the proper choice of a sampling inspection plan can be made only after the risks that can be tolerated have been stated.

If p is equal to the marginal value p', we may say that it is indifferent to us whether the lot is accepted or rejected. If $p < p'$ we prefer acceptance and this preference is the stronger the smaller p. Similarly, if $p > p'$ we prefer rejection of the lot and this preference increases as p increases. Thus, it will be possible to select a value $p_0 < p'$ and a value $p_1 > p'$ such that the error is considered serious only if we accept the lot when $p \geq p_1$, or we reject the lot when $p \leq p_0$.

After the two values p_0 and p_1 have been selected the risks that we are willing to tolerate may reasonably be stated as follows: a sampling inspection plan is required such that the probability of rejecting the lot is less than or equal to a preassigned value α whenever $p \leq p_0$, and the probability of accepting the lot is less than or equal to a preassigned value β whenever $p \geq p_1$. Thus, the tolerated risks are characterized by the four quantities p_0, p_1, α and β. The proper sampling plan can be determined after these four quantities have been chosen.

5.2 3. *The sequential probability ratio test corresponding to the quantities p_0, p_1, α and β.* Let H_0 be the hypothesis that $p = p_0$ and H_1 the hypothesis that $p = p_1$. Consider the sequential probability ratio test T for testing H_0 against H_1 for which α is the probability of accepting H_1 when H_0 is true (error of the first kind) and β is the probability of accepting H_0 when H_1 is true (error of the second kind). This probability ratio test will satisfy all our requirements, since for this test the probability of accepting the lot (accepting H_0) is $\leq \beta$ whenever $p \geq p_1$ and the probability of rejecting the lot (accepting H_1) is $\leq \alpha$ whenever $p \leq p_0$.

According to formulas (3.8), (3.9), (3.10) and section 3.3 the sequential test T is given as follows: At each stage of the inspection, at the m-th observation for each integral value of m, calculate the quantity

$$\frac{p_{1m}}{p_{0m}} = \frac{p_1^{d_m}(1 - p_1)^{m-d_m}}{p_0^{d_m}(1 - p_0)^{m-d_m}}. \qquad (m = 1, 2, \cdots) \tag{5.2}$$

where d_m denotes the number of defectives found in the first m units inspected. Reject the lot (accept H_1) if

$$\frac{p_{1m}}{p_{0m}} \geq \frac{1 - \beta}{\alpha}. \tag{5.3}$$

Accept the lot if

$$\frac{p_{1m}}{p_{0m}} \leq \frac{\beta}{1 - \alpha}. \tag{5.4}$$

Take an additional observation if[14]

$$\frac{\beta}{1-\alpha} < \frac{p_{1m}}{p_{0m}} < \frac{1-\beta}{\alpha}. \tag{5.5}$$

For the purpose of practical computations it is useful to rewrite the inequalities (5.3), (5 4) and (5.5) in a somewhat different form Taking the logarithms of both sides of the inequalities (5.3), (5.4) and (5 5) one can easily verify that these inequalities are equivalent to

$$d_m \geq \frac{\log \frac{1-\beta}{\alpha}}{\log \frac{p_1}{p_0} - \log \frac{1-p_1}{1-p_0}} + m \frac{\log \frac{1-p_0}{1-p_1}}{\log \frac{p_1}{p_0} - \log \frac{1-p_1}{1-p_0}} \tag{5.6}$$

$$d_m \leq \frac{\log \frac{\beta}{1-\alpha}}{\log \frac{p_1}{p_0} - \log \frac{1-p_1}{1-p_0}} + m \frac{\log \frac{1-p_0}{1-p_1}}{\log \frac{p_1}{p_0} - \log \frac{1-p_1}{1-p_0}} \tag{5.7}$$

and

$$\begin{aligned} &\frac{\log \frac{\beta}{1-\alpha}}{\log \frac{p_1}{p_0} - \log \frac{1-p_1}{1-p_0}} + m \frac{\log \frac{1-p_0}{1-p_1}}{\log \frac{p_1}{p_0} - \log \frac{1-p_1}{1-p_0}} \\ &\qquad < d_m < \frac{\log \frac{1-\beta}{\alpha}}{\log \frac{p_1}{p_0} - \log \frac{1-p_1}{1-p_0}} + m \frac{\log \frac{1-p_0}{1-p_1}}{\log \frac{p_1}{p_0} - \log \frac{1-p_1}{1-p_0}}. \end{aligned} \tag{5 8}$$

Using the inequalities (5 6), (5 7) and (5.8) the test procedure can easily be carried out as follows: For each m we compute the acceptance number

$$A_m = \frac{\log \frac{\beta}{1-\alpha}}{\log \frac{p_1}{p_0} - \log \frac{1-p_1}{1-p_0}} + m \frac{\log \frac{1-p_0}{1-p_1}}{\log \frac{p_1}{p_0} - \log \frac{1-p_1}{1-p_0}} \tag{5.9}$$

and the rejection number

$$R_m = \frac{\log \frac{1-\beta}{\alpha}}{\log \frac{p_1}{p_0} - \log \frac{1-p_1}{1-p_0}} + m \frac{\log \frac{1-p_0}{1-p_1}}{\log \frac{p_1}{p_0} - \log \frac{1-p_1}{1-p_0}}. \tag{5.10}$$

[14] There is a slight approximation involved in the formulas (5.3), (5.4) and (5.5). For details see section 3 3

These acceptance numbers A_m and rejection numbers R_m are best tabulated before inspection starts. Inspection is continued as long as $A_m < d_m < R_m$. At the first time when d_m does not lie between the acceptance and rejection numbers, the sampling inspection is terminated. The lot is accepted if $d_m \leq A_m$ and the lot is rejected if $d_m \geq R_m$.

The test procedure can also be carried out graphically as indicated in Figure 2. The number m of observations made is measured along the abscissa axis. Since A_m is a linear function of m, the points (m, A_m) will lie on a straight line L_0. Similarly, the points (m, R_m) will lie on a straight line L_1. We draw the lines L_0 and L_1 and the points (m, d_m) are plotted as inspection goes on. At the first time when the point (m, d_m) does not lie between the lines L_0 and L_1 inspection

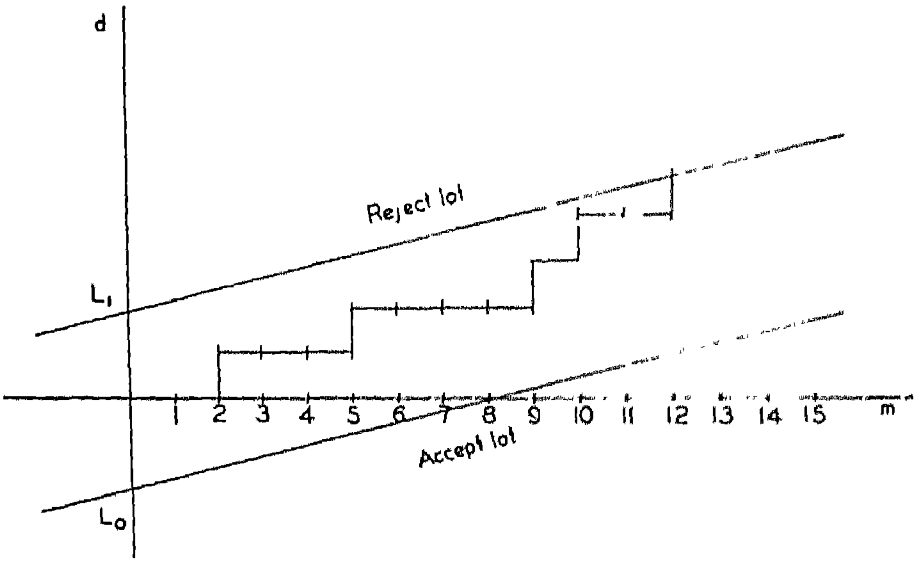

FIG. 2

is terminated. The lot is rejected if the point (m, d_m) lies on L_1 or above, and the lot is accepted if the point (m, d_m) lies on L_0 or below.

5.2.4. *The operating characteristic curve of the test.* As mentioned in section 5.2.3 the test procedure defined by the inequalities (5.6), (5.7) and (5.8) will satisfy the requirement that the probability of accepting the lot is $\leq \beta$ whenever $p \geq p_1$ and the probability of rejecting the lot is $\leq \alpha$ whenever $p \leq p_0$. Although this already describes the essential features of the test procedure, it may be desirable to know the probability L_p of accepting the lot for any possible value p of the proportion of defectives in the lot. Clearly, L_p will be a function of p and can be plotted as shown in Figure 3. The curve L_p is called the operating characteristic curve. The range of p is, of course, from 0 to 1. $L_p = 1$ for $p = 0$ and $L_p = 0$ for $p = 1$. The value of L_p decreases as p increases. We already know that $L_{p_0} = 1 - \alpha$ and $L_{p_1} = \beta$. Now we shall give a method

for computing the value of L_p for any p. If p_1 is not far from p_0, which will usually be the case in practice, a good approximation to L_p is given by (see equation 3.35)

$$(5.11)\qquad L_p \sim 1 - \frac{1 - \left(\frac{\beta}{1-\alpha}\right)^h}{\left(\frac{1-\beta}{\alpha}\right)^h - \left(\frac{\beta}{1-\alpha}\right)^h} = \frac{\left(\frac{1-\beta}{\alpha}\right)^h - 1}{\left(\frac{1-\beta}{\alpha}\right)^h - \left(\frac{\beta}{1-\alpha}\right)^h}.$$

where h is equal to the non-zero root of the equation

$$(5.12)\qquad p\left(\frac{p_1}{p_0}\right)^h + (1-p)\left(\frac{1-p_1}{1-p_0}\right)^h = 1.$$

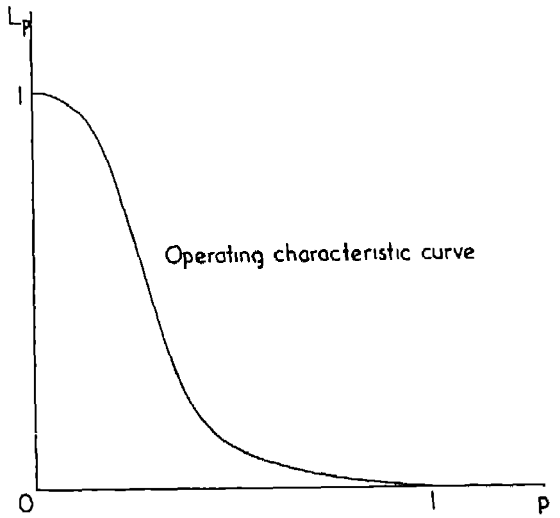

FIG. 3

To plot the operating characteristic curve, it is not necessary to solve (5.12) with respect to h. Instead we can proceed as follows: From (5.12) we express p as a function of h, i.e.,

$$(5.13)\qquad p = \frac{1 - \left(\frac{1-p_1}{1-p_0}\right)^h}{\left(\frac{p_1}{p_0}\right)^h - \left(\frac{1-p_1}{1-p_0}\right)^h}.$$

For any given value h we compute the value of p from (5.13) and the value of L_p from (5.11). The point (p, L_p) obtained in this way will be a point of the operating characteristic curve. Doing this for various values of h we can obtain a sufficient number of points on the operating characteristic curve so that the curve can be drawn.

5.2.5. *The average amount of inspection required by the test.* Denote by $E_p(n)$ the expected value of the number of observations required by the test. Clearly, $E_p(n)$ is a function of p. According to (4.8) a good approximation to the value of $E_p(n)$ is given by

$$E_p(n) \sim \frac{L_p \log \dfrac{\beta}{1-\alpha} + (1 - L_p) \log \dfrac{1-\beta}{\alpha}}{p \log \dfrac{p_1}{p_0} + (1 - p) \log \dfrac{1 - p_1}{1 - p_0}} \tag{5.14}$$

where L_p is given by (5.11). Plotting $E_p(n)$ as a function of p, the curve obtained will, in general, be of the type shown in Fig. 4. The maximum will ordinarily be reached between p_0 and p_1. Furthermore, the curve will, in general, be increasing as p increases from 0 to p_0, and decreasing as p increases from p_1 to 1.

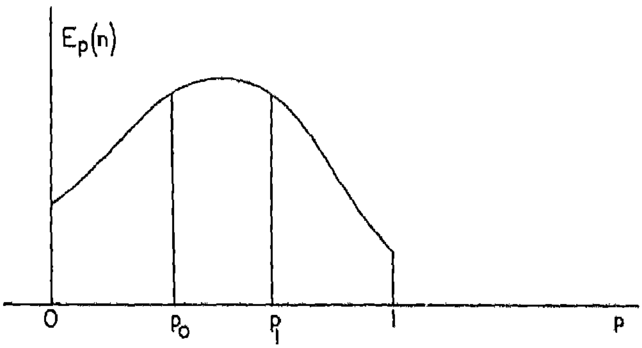

FIG. 4

5.3. *Sequential analysis of double dichotomies.* 5.3.1. *Formulation of the problem.* Suppose that we want to compare the effectiveness of two production processes where the effectiveness of a production process is measured in terms of the proportion of effective units in the sequence produced. We shall say that a unit is effective if it has a certain desirable property, for example, if it withstands a certain strain. Let p_1 be the proportion of effectives if process 1 is used, and p_2 the proportion of effectives if process 2 is used. In other words, p_1 is the probability that a unit produced will be effective if process 1 is used, and p_2 is the probability that a unit produced will be effective if process 2 is used. Suppose that the manufacturer does not know the values of p_1 and p_2, and that process 1 is in operation. If $p_1 \geq p_2$, then the manufacturer wants to retain process 1. However, if $p_1 < p_2$, especially if p_1 is substantially smaller than p_2, the manufacturer would like to replace process 1 by process 2. Thus, we are interested in testing the hypothesis that $p_1 \geq p_2$ against the alternative that $p_1 < p_2$.

A more general formulation of the problem can be given as follows: Consider two binomial distributions. Let p_1 be the probability of a success in a single

trial according to the first binomial distribution, and let p_2 be the probability of a success in a single trial according to the second binomial distribution. We shall use the symbol 1 for success and the symbol 0 for failure. Suppose that the probabilities p_1 and p_2 are unknown. We consider the problem of testing the hypothesis that $p_1 \geq p_2$ on the basis of a sample consisting of N_1 observations from the first binomial distribution and N_2 observations from the second binomial population. Since in many experiments the case $N_1 = N_2$ is mainly of interest, and since this case (as we shall see later) makes an exact and simplified mathematical treatment of the problem possible, we shall assume in what follows that $N_1 = N_2 = N$ (say).

Thus, on the basis of the outcome of the two series of N independent trials we have to decide whether the hypothesis $p_1 \geq p_2$ should be accepted or rejected

5.3.2. *The classical method.* The classical solution of the problem for large N is given as follows: Let S_1 be the number of successes in the first set of N trials (drawn from the first binomial population), and let S_2 be the number of successes in the second set of N trials (drawn from the second binomial population). Denote $\frac{S_1 + S_2}{2N}$ by $\bar{p}$ and $1 - \bar{p}$ by $\bar{q}$. Then for large N the expression

$$\frac{S_2 - S_1}{\sqrt{2N\bar{p}\bar{q}}} \tag{5.15}$$

is normally distributed with zero mean and unit variance if $p_1 = p_2$. Suppose that the level of significance we wish to choose is α. Let λ_α be the value for which the probability that a normal variate with zero mean and unit variance will exceed λ_α is equal to α. (For example, if $\alpha = .05$, $\lambda_\alpha = 1.64$). Thus, if $p_1 = p_2$, the probability that the expression (5.15) will exceed λ_α is equal to α. If $p_1 > p_2$, the probability that the expression (5.15) will exceed λ_α is less than α. According to the classical method the hypothesis that $p_1 \geq p_2$ is rejected if the observed value of (5.15) exceeds λ_α. This method involves an approximation. The distribution of the expression (5.15) is not exactly normal even for large N. For small N this method cannot be used, since the distribution of (5 15) is far from normal. For small N, R. A. Fisher has proposed an exact method which, however, involves cumbersome calculations. In section 5.3.3. we shall suggest another method which is exact (does not involve any approximations) and is simple to apply as far as computations are concerned. The latter method has the further advantage of being suitable for sequential analysis to which existing methods are not readily adaptable.

5 3.3. *An exact method.* Let $a_1, \cdots, a_N$ be the results in the first set of N trials, and $b_1, \cdots, b_N$ the results in the second set of N trials. These results are arranged in the order observed. Consider the sequence of N pairs

$$(a_1, b_1), \cdots, (a_N, b_N). \tag{5.16}$$

Let t_1 be the number of pairs (1, 0) and t_2 the number of pairs (0, 1) in this sequence. We consider only the pairs (0, 1) and (1, 0) and base the test on them.

Let a be the outcome of an observation from the first population, and b the outsome of an observation from the second population. The probability that $(a, b) = (1, 0)$ is equal to $p_1(1 - p_2)$, and the probability that $(a, b) = (0, 1)$ is equal to $(1 - p_1)p_2$. Hence, knowing that (a, b) is equal to one of the pairs $(0, 1)$ and $(1, 0)$, the (conditional) probability that it is equal to $(0, 1)$ is given by

$$p = \frac{(1 - p_1)p_2}{p_1(1 - p_2) + p_2(1 - p_1)}, \tag{5.17}$$

and the (conditional) probability that it is equal to $(1, 0)$ is given by

$$1 - p = \frac{p_1(1 - p_2)}{p_1(1 - p_2) + (1 - p_1)p_2}. \tag{5.18}$$

Hence, considering only the pairs $(1, 0)$ and $(0, 1)$ the variate t_2 is distributed like the number of successes in a sequence of $t = t_1 + t_2$ independent trials, the probability of a success in a single trial being equal to p. One can easily verify that $p = \frac{1}{2}$ if $p_1 = p_2$, $p < \frac{1}{2}$ if $p_1 > p_2$ and $p > \frac{1}{2}$ if $p_1 < p_2$. Thus, the hypothesis to be tested, i.e., the hypothesis that $p_1 \geq p_2$, is equivalent to the hypothesis that $p \leq \frac{1}{2}$. Thus, we can test the hypothesis that $p_1 \geq p_2$ by testing the hypothesis that $p \leq \frac{1}{2}$ on the basis of the observed value of t_2. Since the distribution of t_2 is the same as the distribution of the number of successes in $t = t_1 + t_2$ independent trials (t is treated as a constant and the probability of a success in a single trial is equal to p), the test procedure can be carried out in the usual manner. If we want a level of significance α, a critical value T is chosen so that for $p = \frac{1}{2}$ the probability that $t_2 \geq T$ is equal to α. The hypothesis that $p \leq \frac{1}{2}$ is rejected if and only if the observed t_2 is greater than or equal to the critical value T. The value of T can be obtained from a table of the binomial distribution. If t is large, t_2 is nearly normally distributed and the critical value T can be obtained from a table of the normal distribution.

This procedure thus provides a simple test of the hypothesis that $p_1 \geq p_2$. The question arises whether the efficiency of this method is as high as that of the classical method. It would seem that the method suggested here cannot be a most efficient procedure, since the values of t_1 and t_2 depend on the order of the elements in the sequences $(a_1, \cdots, a_N)$ and $(b_1, \cdots, b_N)$, and there is no particular reason to arrange them in the order observed. However, it has been shown in [7] that the loss in efficiency as compared with the classical method is negligible if the number N of trials is large.[15]

It should be pointed out that the procedure for testing the hypothesis that $p_1 \geq p_2$ can be used also for testing the hypothesis that $p_1 = p_2$ if the alternative hypotheses are restricted to $p_2 > p_1$.

In addition to simplicity and exactness the present method seems superior to the classical one in the following respect: Suppose that (contrary to the original

[15] The author believes that the loss in efficiency is slight even when N is small, although no exact investigation of this case has been made.

assumption) the probability of a success varies from trial to trial. Denote by $p_1^{(i)}$ the probability of success in the i-th trial of the first set, and by $p_2^{(i)}$ the probability of success in the i-th trial in the second set ($i = 1, \cdots, N$). Assume that that the probabilities $p_1^{(i)}$ and $p_2^{(i)}$ are entirely unknown and we wish to test the hypothesis that $p_1^{(1)} - p_2^{(1)} = \cdots = p_1^{(N)} - p_2^{(N)} = 0$. In this case the classical method is not applicable, but the present method provides a correct procedure. Such a situation may arise, for instance, if we want to test the hypothesis that the probability of a success (hitting the target) is the same for two different guns. In the course of the experiments the probability of a hit may change due to external conditions such as wind, disposition of the gunner, etc. However, these external conditions are likely to affect both guns equally if the trials are made alternately (or approximately alternately), so that if the two guns are equally good we have $p_1^{(i)} = p_2^{(i)}$ ($i = 1, \cdots, N$).

5 3.4. *Sequential test of the hypothesis that* $p_1 \geq p_2$. In order to devise a proper sequential test for testing the hypothesis that $p_1 \geq p_2$, we have to state first what risks of making wrong decisions we are willing to tolerate. The efficiency of the production process 1 may be measured by the ratio of effectives to ineffectives produced, i.e., by $k_1 = \dfrac{p_1}{1 - p_1}$. Production process 1 may be regarded the more efficient the larger the value of k_1. Similarly, the efficiency of production process 2 may be measured by $k_2 = \dfrac{p_2}{1 - p_2}$. The relative superiority of production process 2 over the process 1 can then reasonably be measured by the ratio of k_2 to k_1 i.e., by

$$u = \frac{k_2}{k_1}\frac{p_2(1 - p_1)}{p_1(1 - p_2)} \tag{5.19}$$

If $u = 1$, the two processes are equally good. If $u > 1$, process 2 is superior to process 1, and if $u < 1$, process 1 is superior to process 2. Thus, the manufacturer will, in general, be able to select two values of u, u_0 and u_1 say ($u_0 < u_1$) such that the rejection of process 1 in favor of process 2 is considered an error of practical importance whenever the true value of $u \leq u_0$, and the maintainance of process 1 is considered an error of practical importance whenever $u \geq u_1$. If u lies between u_0 and u_1, the manufacturer does not care particularly which decision is taken.

Clearly, we will always have $u_0 < u_1$. If the transition from production process 1 to process 2 involves some cost or other inconveniences, it seems reasonable to put $u_0 = 1$ (or u_0 may even be slightly greater than one). This choice of u_0 really means that we consider the rejection of process 1 a serious error whenever this process is not inferior to process 2. On the other hand, if the transition from process 1 to process 2 does not involve any inconveniences, the rejection of process 1 in favor of 2 cannot be a serious error when the two processes are equally efficient, i.e., when $u = 1$. Thus, in such a case, it seems reasonable to choose u_0 somewhat below 1.

After the quantities u_0 and u_1 have been chosen the risks that we are willing to tolerate may reasonably be expressed in the following form: The probability of rejecting process 1 should not exceed a preassigned value α whenever $u \leq u_0$, and the probability of maintaining process 1 should not exceed a preassigned value β whenever $u \geq u_1$.

Thus, the risks that we are willing to tolerate are characterized by the four quantities u_0, u_1, α and β. After these four quantities have been chosen, a proper sequential test can be carried out as follows: The (conditional) probability that we obtain a pair (0, 1), as given in (5.17), can be expressed as a function of u. In fact

$$(5.20) \qquad p = \frac{(1-p_1)p_2}{p_1(1-p_2)+p_2(1-p_1)} = \frac{\dfrac{(1-p_1)p_2}{p_1(1-p_2)}}{1+\dfrac{p_2(1-p_1)}{p_1(1-p_2)}} = \frac{u}{1+u}.$$

Let H_0 denote the hypothesis that $p = \dfrac{u_0}{1+u_0}$, and H_1 the hypothesis that $p = \dfrac{u_1}{1+u_1}$. A proper sequential test satisfying our requirements concerning tolerated risks is the sequential probability ratio test of H_0 against H_1. The acceptance and rejection numbers for this sequential test can be obtained from (5.9) and (5.10) by substituting $\dfrac{u_0}{1+u_0}$ for p_0 $\dfrac{u_1}{1+u_1}$ for p_1 and $t = t_1 + t_2$ for m.

Thus, for each value of t the acceptance number is given by

$$(5.21) \qquad A_t = \frac{\log \dfrac{\beta}{1-\alpha}}{\log u_1 - \log u_0} + t\,\frac{\log \dfrac{1+u_1}{1+u_0}}{\log u_1 - \log u_0}$$

and the rejection number is given by

$$(5.22) \qquad R_t = \frac{\log \dfrac{1-\beta}{\alpha}}{\log u_1 - \log u_0} + t\,\frac{\log \dfrac{1+u_1}{1+u_0}}{\log u_1 - \log u_0}$$

These acceptance numbers A_t and rejection numbers $R_t (t = 1, 2, \cdots)$ are best tabulated before experimentation starts. The sequential test is then carried out as follows: The observations are taken in pairs where each pair consists of an observation from the first process and an observation from the second process. We continue taking pairs as long as $A_t < t_2 < R_t$. At the first time when t_2 does not lie between the acceptance and rejection numbers, experimentation is terminated. Process 1 is maintained if at this final stage $t_2 \leq A_t$, and process 1 is rejected in favor of 2 if $t_2 \geq R_t$.

The test procedure can also be carried out graphically as shown in Figure 5. The total number m of pairs (0, 1) and (1, 0) is measured along the horizontal axis. The points (t, A_t) will lie on a straight line L_0, since A_t is a linear function of t. The points (t, R_t) will lie on a parallel line L_1. We draw the lines L_0 and

L_1 and plot the points (t, t_2) as experimentation goes on At the first time when the point (t, t_2) is not within the lines L_0 and L_1 experimentation is terminated. Process 1 is maintained if at the final stage the point (t, t_2) lies on L_0 or below, and process 1 is rejected if the point (t, t_2) lies on L_1 or above.

5.3.5. *The operating characteristic curve of the test.* For any value u of the ratio $\frac{k_2}{k_1}$ we shall denote by L_u the probability of maintaining process 1. Clearly, L_u is a function of u. This function L_u is called the operating characteristic curve of the test. The operating characteristic curve can be determined from the equations (5.11) and (5.13) by substituting $\frac{u_1}{1 + u_1}$ for p_1 and $\frac{u_0}{1 + u_0}$ for p_0.

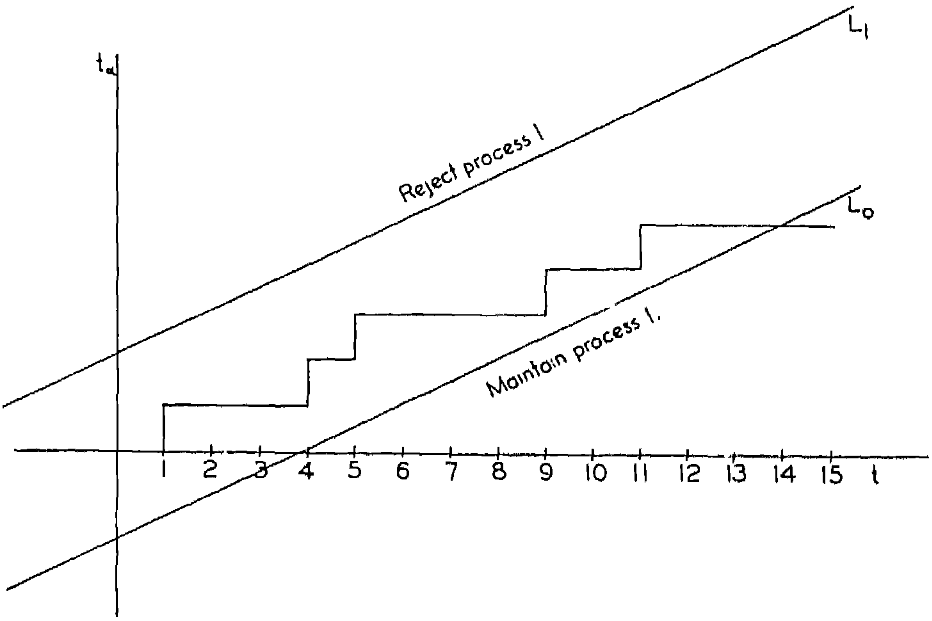

FIG. 5

These equations are:

$$L_u \sim \frac{\left(\frac{1-\beta}{\alpha}\right)^h - 1}{\left(\frac{1-\beta}{\alpha}\right)^h - \left(\frac{\beta}{1-\alpha}\right)^h} \tag{5.23}$$

and

$$\frac{u}{1+u} = \frac{1 - \left(\frac{1+u_0}{1+u_1}\right)^h}{\left(\frac{u_1(1+u_0)}{u_0(1+u_1)}\right)^h - \left(\frac{1+u_0}{1+u_1}\right)^h}. \tag{5.24}$$

For any given value h we compute the values of u and L_u from these equations. The point (u, L_u) obtained in this way will be a point of the operating characteristic curve. Calculating the points (u, L_u) for a sufficiently large number of values of h we can draw the operating characteristic curve.

5.3.6. *The average amount of inspection required by the test.* For any value u of the ratio $\frac{k_2}{k_1}$ denote by $E_u(t)$ the expected value of the total number of pairs $(0, 1)$ and $(1, 0)$ required by the test. The value of $E_u(t)$ can be obtained from (5.14) by substituting $E_u(t)$ for $E_p(n)$, L_u for L_p, $\frac{u_1}{1 + u_1}$ for p_1 and $\frac{u_0}{1 + u_0}$ for p_0. Thus

$$E_u(t) \sim \frac{L_u \log \frac{\beta}{1 - \alpha} + (1 - L_u) \log \frac{1 - \beta}{\alpha}}{\frac{u}{1 + u} \log \frac{u_1(1 + u_0)}{u_0(1 + u_1)} - \frac{1}{1 + u} \log \frac{1 + u_0}{1 + u_1}}. \tag{5.25}$$

To compute the expected value of the total number of pairs (including also the pairs $(0, 0)$ and $(1, 1)$), we merely have to divide the right side expression in (5.25) by $p_1(1 - p_2) + p_2(1 - p_1)$.

In the rare event that no decision is yet reached at a number of pairs equal to three times the expected value, we can truncate the test at that stage without seriously affecting the probabilities of making a wrong decision (see section 4.6 in Part I).

5.3.7. *Observations made in groups of r.* In applications it may happen that at each stage in the sequential process instead of drawing a single observation we draw r observations from each of the binomial distributions. Hence, instead of a single pair, we have two sets of r observations. If the order of observations in each such set of r is recorded, we can establish the number of pairs $(0, 1)$ and the number of pairs $(1, 0)$ for each pair of sets of r observations. In such a case the test can be carried out as described in section 5.3.4, since after each pair of sets of r observations we can compute t and t_2. The only effect of taking the observations in groups of r is that more observations will generally be necessary (approximately enough to fill out a group) and thereby the probability of making an incorrect decision will be made somewhat smaller. However, if the order of observations in such groups of r is not recorded, the difficulty arises that we are not able to determine the values of t and t_2 needed for the test procedure. It has been shown in [7] that in such a case we may replace t and t_2 by certain estimates of t and t_2 without affecting seriously the probability of making an incorrect decision. The estimates of t_1 and t_2 (and thereby also an estimate of $t = t_1 + t_2$) are obtained as follows: Let r_1 be the number of successes in the group of r observations drawn from the first binomial distribution, and let r_2 be the number of successes in the group of r observations drawn from the second binomial distribution. Then for this pair of groups of r observations, we estimate the number

of pairs (1, 0) to be $r_1 - \frac{r_1 r_2}{r}$ and the number of pairs (0, 1) to be $r_2 - \frac{r_1 r_2}{r}$. Thus, an estimate of t_1 is obtained by summing $r_1 - \frac{r_1 r_2}{r}$ over all pairs of groups observed, and that of t_2 is obtained by summing $r_2 - \frac{r_1 r_2}{r}$ over all pairs of groups observed.

5.4. *Application to testing the mean of a normal distribution with known standard deviation.* 5.4.1. *Formulation of the problem.* Suppose that a measurable quantity x is normally distributed with unknown mean θ and known standard deviation σ. For example, x may be some measurable quality characteristic of a unit of a certain product where x is normally distributed with a known standard deviation in the population of all units. The problem we shall consider here is to test the hypothesis that the unknown mean θ is less than a specified value θ'. This problem arises frequently, for example, in quality control. Suppose that the quality of the product is considered the better the higher the mean value of x. Thus, there will be a value θ' such that the product is considered sub-standard if $\theta < \theta'$ and the product is considered to meet specifications if $\theta \geq \theta'$. Since θ is unknown, we are usually interested in testing the hypothesis that $\theta < \theta'$, i.e., that the product is sub-standard.

Since quality control is an important field of application for such test procedures, the discussion will be continued in the terminology of quality control. This, of course, should not be interpreted as a restriction upon the general validity and applicability of the test procedure. The problem treated in section 5.4 can now be stated as follows: Let x be a measurable quality characteristic of a unit of a certain product. The variable x is supposed to be normally distributed with known standard deviation in the population of all units produced. The problem is to devise a sampling plan for testing the hypothesis that the product is sub-standard. The product is said to be sub-standard, if the mean θ of x is less than a given specified value θ'.

5.4.2. *Tolerated risks for making a wrong decision.* No sampling plan can guarantee that the correct decision will always be made, i.e., that the product will be declared sub-standard if and only if $\theta < \theta'$. The larger the amount of inspection, the smaller we can make the risks for making a wrong decision. If inspection is costly, or destructive, we are willing to tolerate some risks of making wrong decisions in order to reduce the necessary amount of inspection. Thus, a proper sampling plan can be recommended only after the risks that can be tolerated have been stated.

If the quality of the product is exactly on the margin, i.e., if $\theta = \theta'$, then it will make little difference whether the product is classified as sub-standard or not. However, if θ is considerably smaller than θ', then the acceptance of the hypothesis that the product meets specifications (rejection of the hypothesis that the product is sub-standard) will usually be considered as a serious error.

Similarly, if θ is much larger than θ', the acceptance of the hypothesis that the product is sub-standard will generally be considered as a serious error. Thus, the manufacturer will, in general, be able to select two values of θ, θ_0 and θ_1 say ($\theta_0 < \theta'$ and $\theta_1 > \theta'$) such that the classification of the product as satisfactory (meeting specifications) is considered an error of practical importance whenever $\theta \leq \theta_0$, and the classification of the product as sub-standard is considered an error of practical importance whenever $\theta \geq \theta_1$. If θ lies between θ_0 and θ_1, a wrong classification of the product will not be viewed as a serious error, since in this case θ is near the marginal value θ'.

After the two values θ_0 and θ_1 have been selected, the risks that we are willing to tolerate can be stated in the following form: A sampling plan is required such that the probability of classifying the product as satisfactory is less than or equal to a preassigned quantity α whenever $\theta \leq \theta_0$, and such that the probability of classifying the product as sub-standard is less than or equal to a preassigned quantity β whenever $\theta \geq \theta_1$. Thus, the tolerated risks are characterized by the four quantities θ_0, θ_1, α and β. A proper sampling plan can be devised after these four quantities have been selected.

5.4.3. *A sequential test of the hypothesis that $\theta < \theta'$ (the product is sub-standard).* Let H_0 be the hypothesis that $\theta = \theta_0$ and let H_1 be the hypothesis that $\theta = \theta_1$. Let T be the sequential probability ratio test for testing H_0 against H_1 such that α is the probability of accepting H_1 when H_0 is true and β is the probability of accepting H_0 when H_1 is true. This sequential test will satisfy all our requirements, since for this test the probability of accepting H_0 (declaring the product as sub-standard) is $\leq \beta$ whenever $\theta \geq \theta_1$, and the probability of accepting H_1 (declaring the product as satisfactory) is $\leq \alpha$ whenever $\theta \leq \theta_0$.

The sequential test T is given as follows: Denote the successive observations on x by $x_1, x_2, \cdots$, etc. Accept the hypothesis that the product is satisfactory at the m-th observation if

$$\log \frac{e^{-(1/2\sigma^2)\sum_{\alpha=1}^{m}(x_\alpha-\theta_1)^2}}{e^{-(1/2\sigma^2)\sum_{\alpha=1}^{m}(x_\alpha-\theta_0)^2}} \geq \log \frac{1-\beta}{\alpha}. \tag{5.26}$$

Accept the hypothesis that the product is sub-standard if

$$\log \frac{e^{-(1/2\sigma^2)\sum_{\alpha=1}^{m}(x_\alpha-\theta_1)^2}}{e^{-(1/2\sigma^2)\sum_{\alpha=1}^{m}(x_\alpha-\theta_0)^2}} \leq \log \frac{\beta}{1-\alpha} \tag{5.27}$$

Take an additional observation if

$$\log \frac{\beta}{1-\alpha} < \log \frac{e^{-(1/2\sigma^2)\sum_{1}^{m}(x_\alpha-\theta_1)^2}}{e^{-(1/2\sigma^2)\sum_{1}^{m}(x_\alpha-\theta_0)^2}} < \log \frac{1-\beta}{\alpha}. \tag{5.28}$$

The inequalities (5.26), (5.27) and (5.28) are equivalent to

$$\sum_{\alpha=1}^{m} x_\alpha \geq \frac{\sigma^2}{\theta_1 - \theta_0} \log \frac{1-\beta}{\alpha} + m \frac{\theta_0 + \theta_1}{2} \tag{5.29}$$

$$\sum_{\alpha=1}^{m} x_\alpha \leq \frac{\sigma^2}{\theta_1 - \theta_0} \log \frac{\beta}{1-\alpha} + m \frac{\theta_0 + \theta_1}{2} \tag{5.30}$$

and

$$\frac{\sigma^2}{\theta_1 - \theta_0} \log \frac{\beta}{1-\alpha} + m \frac{\theta_0 + \theta_1}{2} < \sum_{\alpha=1}^{m} x_\alpha < \frac{\sigma^2}{\theta_1 - \theta_0} \log \frac{1-\beta}{\alpha} + m \frac{\theta_0 + \theta_1}{2}, \tag{5.31}$$

respectively.

Using the inequalities (5.29), (5.30) and (5.31) the test procedure can easily be carried out as follows: For each m compute the acceptance number

$$A_m = \frac{\sigma^2}{\theta_1 - \theta_0} \log \frac{\beta}{1-\alpha} + m \frac{\theta_0 + \theta_1}{2} \tag{5.32}$$

and the rejection number

$$R_m = \frac{\sigma^2}{\theta_1 - \theta_0} \log \frac{1-\beta}{\alpha} + m \frac{\theta_0 + \theta_1}{2}. \tag{5.33}$$

These acceptance numbers A_m and rejection numbers R_m are best tabulated before inspection starts. Inspection is continued as long as $A_m < \sum_{\alpha=1}^{m} x_\alpha < R_m$. At the first time that $\sum_{\alpha=1}^{m} x_\alpha$ does not lie between A_m and R_m, inspection is terminated. If at this final stage $\sum_{\alpha=1}^{m} x_\alpha \leq A_m$, the hypothesis that the product is sub-standard is accepted, and if $\sum_{\alpha=1}^{m} x_\alpha \geq R_m$, the hypothesis that the product is sub-standard is rejected

The test procedure can also be carried out graphically as shown in Figure 6. The number m of observations is measured along the horizontal axis. The points (m, A_m) will lie in a straight line L_0 and the points (m, R_m) will lie on a parallel line L_1. We draw the parallel lines L_0 and L_1 and plot the points $\left(m, \sum_{\alpha=1}^{m} x_\alpha\right)$ as inspection goes on. At the first time when the point $\left(m, \sum_{\alpha=1}^{m} x_\alpha\right)$ does not lie between the lines L_0 and L_1 inspection is terminated. The hypothesis that the product is sub-standard is rejected if the point $\left(m, \sum_{\alpha=1}^{m} x_\alpha\right)$ lies on L_1 or above. The hypothesis in question is accepted if the point $\left(m, \sum_{\alpha=1}^{m} x_\alpha\right)$ lies on L_0 or below.

5.4.4. *The operating characteristic curve of the test.* For any value θ denote by L_θ the probability that the hypothesis that the product is sub-standard is accepted. Obviously, L_θ will be a function of θ and is called the operating characteristic curve of the test. The shape of the operating characteristic curve will, in general, be of the type shown in Figure 7. L_θ approaches 1 as $\theta \to -\infty$ and L_θ approaches zero as $\theta \to \infty$. Furthermore, L_θ is a decreasing function of θ. We already know the values of L_θ for $\theta = \theta_0$ and $\theta = \theta_1$. Now we shall give a method for computing the value of L_θ for any θ. If $\frac{\theta_1 - \theta_0}{\sigma}$ is fairly small,

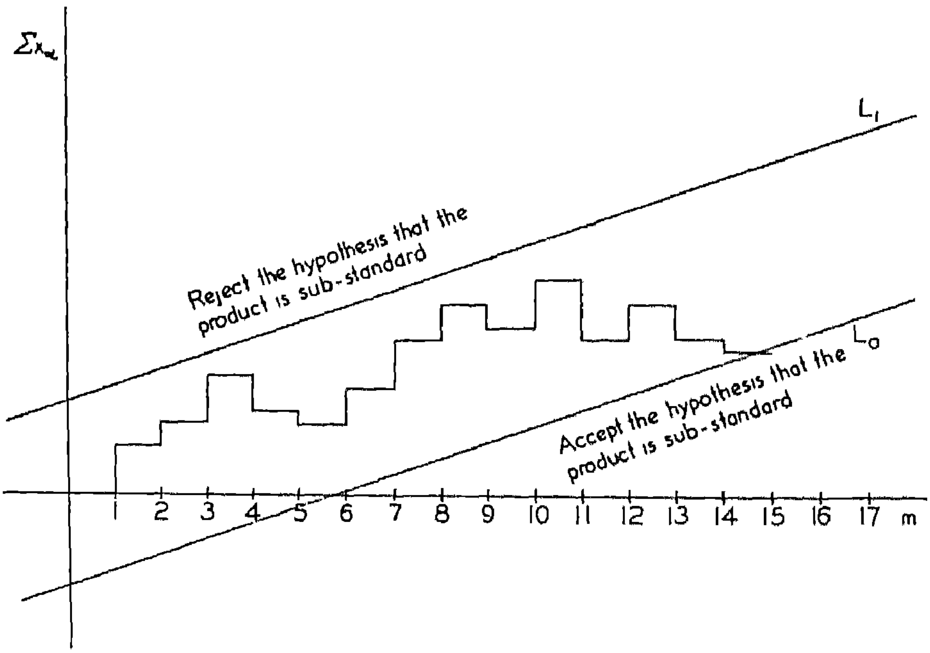

FIG. 6

which will usually be the case in practice, a good approximation to L_θ is given by (see equation 3.35)

$$(5.34) \qquad L_\theta \sim 1 - \frac{1 - \left(\frac{\beta}{1-\alpha}\right)^h}{\left(\frac{1-\beta}{\alpha}\right)^h - \left(\frac{\beta}{1-\alpha}\right)^h} = \frac{\left(\frac{1-\beta}{\alpha}\right)^h - 1}{\left(\frac{1-\beta}{\alpha}\right)^h - \left(\frac{\beta}{1-\alpha}\right)^h}$$

where the constant h is determined as follows: First we compute the characteristic function $\varphi(t)$ of the variate

$$(5.35) \qquad z = \log \frac{e^{-\frac{1}{2\sigma^2}(x-\theta_1)^2}}{e^{-\frac{1}{2\sigma^2}(x-\theta_0)^2}} = \frac{1}{2\sigma^2}\left[2(\theta_1 - \theta_0)x + \theta_0^2 - \theta_1^2\right].$$

Thus, z is normally distributed with mean $= \dfrac{\theta_0^2 - \theta_1^2}{2\sigma^2} + \dfrac{(\theta_1 - \theta_0)\theta}{\sigma^2}$ and variance $= \dfrac{(\theta_1 - \theta_0)^2}{\sigma^2}$. Consequently, $\varphi(t)$ is given by

$$\varphi(t) = e^{\left[\frac{\theta_0^2-\theta_1^2}{2\sigma^2}+\frac{(\theta_1-\theta_0)\theta}{\sigma^2}\right]t+\frac{(\theta_1-\theta_0)^2}{2\sigma^2}t^2}. \tag{5.36}$$

The value h is the non-zero real root of the equation $\varphi(t) = 1$. Hence

$$h = \frac{(\theta_1^2 - \theta_0^2) - 2(\theta_1 - \theta_0)\theta}{(\theta_1 - \theta_0)^2} = \frac{\theta_1 + \theta_0 - 2\theta}{\theta_1 - \theta_0} \tag{5.37}$$

The operating characteristic curve can be computed from (5.34) substituting the right hand side member of (5.37) for h.

5.4.5. *The average amount of inspection required by the test.* Let $E_\theta(n)$ denote the expected value of the number of observations required by the test when θ

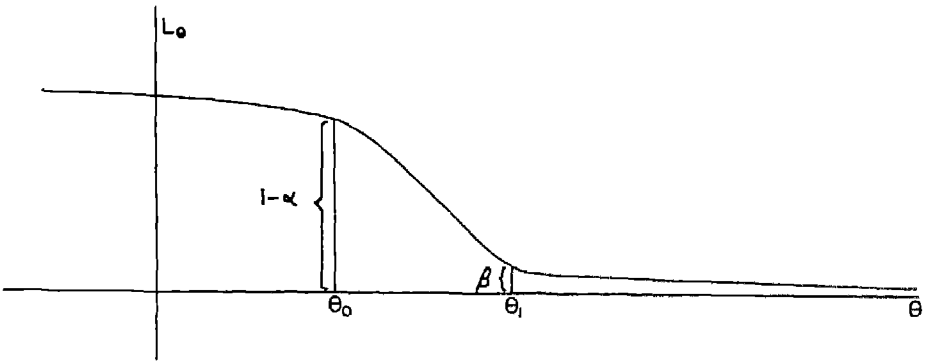

FIG. 7

is the true mean of x. According to (4.8) a good approximation to the value of $E_\theta(n)$ is given by

$$E_\theta(n) \sim 2\sigma^2 \frac{L_\theta \log \dfrac{\beta}{1-\alpha} + (1 - L_\theta) \log \dfrac{1-\beta}{\alpha}}{\theta_0^2 - \theta_1^2 + 2(\theta_1 - \theta_0)\theta}$$

where L_θ is given by (5.34)

In the rare event that the number of observations reaches three times the expected value before the test is terminated, we can truncate the test at this stage without seriously affecting the probabilities of making a wrong decision. (See section 4.6 in Part I).

6. Outline of a General Theory of Sequential Tests of Hypotheses when No Restrictions Are Imposed on the Alternative Values of the Unknown Parameters

6.1. *Sequential test of a simple hypothesis with no restrictions on the alternative values of the unknown parameters.* Consider the following general case. Let

$X_1, \cdots, X_p$ be a set of p random variables and let $f(x_1, \cdots, x_p, \theta_1, \cdots, \theta_k)$ be the joint probability density function of these random variables involving k unknown parameters $\theta_1, \cdots, \theta_k$. Suppose that we wish to test the hypothesis H_0 that $\theta_1 = \theta_1^0, \cdots, \theta_k = \theta_k^0$, where $\theta_1^0, \cdots, \theta_k^0$ are some given specified values. Denote the set of all *a priori* possible parameter points by Ω. Assume that Ω contains at least a finite k-dimensional sphere with the center $(\theta_1^0, \cdots, \theta_k^0)$. Let Ω^* be the set of all possible alternative parameter points; i.e., Ω^* is the whole parameter space Ω with the exception of the point $\theta^0 = (\theta_1^0, \cdots, \theta_k^0)$. For any statistical procedure for testing H_0, the probability of an error of the first kind, will have a definite value, but the probability of an error of the second kind will depend on the true alternative; i.e., it will be a single valued function $\beta(\theta)$ defined over all points θ of Ω^*. Let $w(\theta)$ be some non-negative function, called weight function, such that $\int_{\Omega^*} w(\theta)\, d\theta = 1$. Suppose that we wish to construct a sequential test such that the probability of an error of the first kind is equal to a given α and that the weighted average $\int_{\Omega^*} w(\theta)\beta(\theta)\, d(\theta)$ of the probabilities of errors of the second kind is equal to some given positive value β.

This problem can easily be solved as follows: Let p_{0n} be equal to the product $\prod_{\alpha=1}^{n} f(x_{1\alpha}, \cdots, x_{p\alpha}, \theta_1^0, \cdots, \theta_k^0)$ where $x_{i\alpha}$ denotes the αth observation on x_i $(i = 1, \cdots, p; \alpha = 1, \cdots, n)$. Furthermore, let p_{1n} be defined by

$$p_{1n} = \int_{\Omega^*} w(\theta) \left[\prod_{\alpha=1}^{n} f(x_{1\alpha}, \ldots, x_{p\alpha}, \quad \theta_1, \ldots, \theta_k) \right] d\theta. \tag{6.1}$$

The expression p_{1n} can be interpreted as the probability density in the sample space of n observations on the variates $x_1, \cdots, x_p$, if we assume that the parameter point θ in Ω^* has a probability distribution given by the density function $w(\theta)\, d\theta$

We shall denote by H_1 the hypothesis that the probability density function in the sample space of n observations on $X_1, \cdots, X_p$ is given by p_{1n} defined in equation (6 1). The problem of testing H_0 against the single alternative H_1 is not exactly of the type discussed in Part I, since p_{1n} given in (6.1) cannot be represented, in general, as a product of n factors where the αth factor depends only on the observations $x_{1\alpha}, \cdots, x_{p\alpha}$. However, it was pointed out in section 3.2 that the fundamental inequalities derived in Section 3.2 remain valid also when p_{1n} is given by an expression of the type (6.1). Thus, we can use the sequential probability ratio test for testing H_0 against the single alternative H_1. We reject H_0 if

$$\frac{p_{1n}}{p_{0n}} \geq A, \tag{6.2}$$

we accept H_0 if

$$\frac{p_{1n}}{p_{0n}} \leq B, \tag{6.3}$$

and we make an additional observation if

$$B < \frac{p_{1n}}{p_{0n}} < A. \tag{6.4}$$

The expression p_{1n} is given by (6.1) and the constants A and B are chosen so that the probability of accepting H_1 when H_0 is true is α, and the probability of accepting H_0 when H_1 is true is β. Thus, for practical purposes we may put $A = \frac{1 - \beta}{\alpha}$ and $B = \frac{\beta}{1 - \alpha}$.

Using the sequential process defined by the inequalities (6.2), (6.3), and (6.4) we obviously have

$$\int_{\Omega^*} w(\theta)\,\beta(\theta)\,d\theta = \beta \tag{6.5}$$

where for each point θ in Ω^*, $\beta(\theta)$ denotes the probability of accepting H_0 under the assumption that θ is the true parameter point

Thus, the sequential test given by (6.2), (6.3), and (6.4) provides a satisfactory solution of the problem if we want a test procedure such that the probability of an error of the first kind is α and the weighted average $\int_{\Omega^*} w(\theta)\beta(\theta)\,d\theta$ of the probabilities of errors of the second kind is β. Practical problems, however, do not always take this form. Many instances require a test procedure such that $\beta(\theta)$ should be less than or equal to a given positive value β for all parameter points θ whose "distance" (defined in some sense) from θ^0 is greater than or equal to some given positive value d_0. The "distance" of two parameter points θ^1 and θ^2 may be defined by some function $\delta(\theta^1, \theta^2)$ which is equal to zero if $\theta^1 = \theta^2$ and is greater than zero if $\theta^1 \neq \theta^2$. Furthermore, for any three points θ^1, θ^2, θ^3 we have $\delta(\theta^1, \theta^2) = \delta(\theta^2, \theta^1)$ and $\delta(\theta^1, \theta^2) + \delta(\theta^3, \theta^2) \geq \delta(\theta^1, \theta^3)$. The distance function will, in general, be chosen according to practical needs and mathematical convenience.

Given the distance function $\delta(\theta^1, \theta^2)$ and given the requirements that the probability of an error of the first kind be α and the probability of an error of the second kind should not exceed β whenever the distance of the true parameter point from θ^0 is greater than or equal to d_0, the aim is, of course, to construct a sequential test which satisfies these requirements with a minimum expected number of observations.

While an exact solution of this problem has not yet been found, the following approach seems reasonable: Let Ω_0 be the set of all parameter points θ for which $\delta(\theta^0, \theta) \geq d_0$. We restrict ourselves to the class C_S of sequential tests based on the ratio $\frac{p_{1n}}{p_{0n}}$ where

$$p_{0n} = \prod_{\alpha=1}^{n} f(x_{1\alpha}, \cdots, x_{p\alpha}, \quad \theta_1^0, \cdots, \theta_k^0), \tag{6.6}$$

$$p_{1n} = \int_{\Omega_0} w(\theta) \prod_{\alpha=1}^{n} f(x_{1\alpha}, \cdots, x_{p\alpha}, \quad \theta_1, \cdots, \theta_k)\,d\theta \tag{6.7}$$

and $w(\theta)$ may be any non-negative function of θ, called weight function, for which

$$\int_{\Omega_0} w(\theta)d\theta = 1. \tag{6.8}$$

For carrying out the sequential test two constants A and B are chosen. The hypothesis H_0 is accepted if $\frac{p_{1n}}{p_{0n}} \leq B$, H_0 is rejected if $\frac{p_{1n}}{p_{0n}} \geq A$, and an additional observation is made if $B < \frac{p_{1n}}{p_{0n}} < A$. The restriction to the class C_S of sequential tests is suggested by the fact that we are led to these tests if it is require" that some weighted average of the probabilities of errors of the second kind be equal to a given value β.

Accepting the restriction that the sequential test should be a member of the class C_S, we still need a principle for choosing the weight function $w(\theta)$. It is clear that the maximum of $\beta(\theta)$ in Ω_0 depends on the quantities A, B, and the weight function $w(\theta)$. Denote this maximum value by $\beta_{\text{Max}}[A, B, w(\theta)]$. Since it is desirable to make $\beta_{\text{Max}}[A, B, w(\theta)]$ as small as possible, it is proposed to determine $w(\theta)$ so that the expression $\beta_{\text{Max}}[A, B, w(\theta)]$ becomes a minimum with respect to $w(\theta)$. Since for given values A and B the value of the weighted average $\int_{\Omega_0} w(\theta)\beta(\theta)\, d\theta$ is practically independent of $w(\theta)$ (it is nearly equal to $\frac{B(A-1)}{A-B}$), minimizing $\beta_{\text{Max}}[A, B, w(\theta)]$ is practically equivalent to minimizing the difference $\beta_{\text{Max}}[A, B, w(\theta)] - \int_{\Omega_0} w(\theta)\beta(\theta)\, d\theta$. For convenience we determine $w(\theta)$ so that $\beta_{\text{Max}}[A, B, w(\theta)] - \int_{\Omega_0} w(\theta)\beta(\theta)\, d\theta$ becomes a minimum. For this weight function the maximum of $\beta(\theta)$ in Ω_0 will depend only on A and B. Denote this value by $\beta(A, B)$. Finally we determine the values A and B so that $\beta(A, B) = \beta$ and the probability of an error of the first kind becomes α.

The determination of $w(\theta)$ is a problem in the calculus of variations. In some important cases, however, the solution can be obtained by the following simple procedure: Let $S(d)$ be the set of all parameter points θ for which $\delta(\theta^0, \theta) = d$. Let $v(\theta)$ be a non-negative weight function defined over the surface $S(d_0)$ so that the surface integral $\int_{S(d_0)} v(\theta)\, d\omega = 1$ (where $d\omega$ denotes the infinitesimal surface element). Consider the following sequential procedure: Reject H_0 if

$$\frac{\int_{S(d_0)} v(\theta) \left[\prod_\alpha f(x_{1\alpha}, \cdots, x_{p\alpha}, \quad \theta_1, \cdots, \theta_k)\right] d\omega}{\prod_\alpha f(x_{1\alpha}, \cdots, x_{p\alpha}, \quad \theta_1^0, \cdots, \theta_k^0)} \tag{6.9}$$

is greater than or equal to A, accept H_0 if (6.9) is less than or equal to B, and make an additional observation if the value of (6.9) lies between A and B. The

constants A and B are so chosen that the probability of an error of the first kind is α and $\int_{S(d_0)} \beta(\theta)v(\theta)\, d\omega = \beta$. In many statistical problems it is possible to find a weight function $v(\theta)$ such that for a conveniently chosen distance function $\delta(\theta^1, \theta^2)$ the probability $\beta(\theta)$ of an error of the second kind becomes constant on the surface $S(d)$ for any value d, and, furthermore, $\beta(\theta)$ decreases with increasing d. For such a weight function $v(\theta)$, the sequential test based on (6.9), will provide a solution of the problem. In fact, the weight function $v(\theta)$ over the surface $S(d_0)$ can be considered a limiting case of a weight function $w(\theta)$ defined in Ω_0 which takes the value zero for any θ whose distance from θ^0 is greater than $d_0 + \Delta$ with Δ approaching zero in the limit. For the weight function $v(\theta)$ the maximum of $\beta(\theta)$ in Ω_0 is equal to the weighted integral of $\beta(\theta)$. Thus, for this weight function the difference between the maximum of $\beta(\theta)$ and the weighted integral of $\beta(\theta)$ is minimized.

We shall illustrate this procedure by a simple example. Let $X_1, \cdots, X_k$ be k normally and independently distributed variates with unit variances. The mean values $\theta_1, \cdots, \theta_k$ are unknown. Suppose that it is required to test the hypothesis H_0 that $\theta_1 = \cdots = \theta_k = 0$. Assume that the distance of two points θ^1 and θ^2 is equal to

$$+\sqrt{(\theta_1^1 - \theta_1^2)^2 + \cdots + (\theta_k^1 - \theta_k^2)^2}.$$

Then $S(d)$ is a sphere with center at the origin and radius d. Let $v(\theta)$ be constant on $S(d_0)$ and equal to the reciprocal of the area of $S(d_0)$. We shall show that for this weight function $v(\theta)$, $\beta(\theta)$ is constant on the sphere $S(d)$ and is monotonically decreasing with increasing d. For this purpose we prove first that (6.9) is a monotonically increasing function of $\bar{x}_1^2 + \cdots + \bar{x}_k^2$ where $\bar{x}_i$ is the arithmetic mean of the observations on x_i. In fact, the expression (6.9) becomes

$$(6.10)\qquad \frac{c_k \dfrac{1}{(2\pi)^{kn/2}} \displaystyle\int_{S(d_0)} \exp\left[-\frac{1}{2}\sum_{i=1}^{k}\sum_{\alpha=1}^{n}(x_{i\alpha} - \theta_i)^2\right] d\omega}{\dfrac{1}{(2\pi)^{kn/2}} \exp\left[-\frac{1}{2}\Sigma\Sigma x_{i\alpha}^2\right]}$$

$$= c_k \exp\left[-\tfrac{1}{2} n d_0^2\right] \int_{S(d_0)} \exp\left[n\Sigma \bar{x}_i \theta_i\right] d\omega$$

where c_k is the reciprocal of the area of $S(d_0)$ and $\bar{x}_i$ is the arithmetic mean of the n observations $x_{i\alpha}$ $(\alpha = 1, \cdots, n)$. Let r_x denote $\left|\sqrt{\sum_i \bar{x}_i^2}\right|$ and let $\alpha(\theta)$ $(0 \leq \alpha \leq \pi)$ denote the angle between the vector $(\bar{x}_1, \cdots, \bar{x}_k)$ and the vector $(\theta_1, \cdots, \theta_k)$. Then (6.10) can be written

$$(6.11)\qquad c_k \exp\left[-\tfrac{1}{2} n d_0^2\right] \int_{S(d_0)} \exp\left(n r_x d_0 \cos[\alpha(\theta)]\right) d\omega.$$

Because of the symmetry of the sphere, the value of (6.11) will not be changed if we substitute $\gamma(\theta)$ for $\alpha(\theta)$ where $\gamma(\theta)$ $(0 \leq \gamma(\theta) \leq \pi)$ denotes the angle

between the vector θ and an arbitrarily chosen fixed vector u. From this it follows that the value of (6.11) depends only on r_x.

Now we shall show that (6.11) is a strictly increasing function of r_x. For this purpose we have merely to show that

$$(6.12) \qquad I(r_x) = \int_{S(d_0)} \exp (nr_x d_0 \cos [\gamma(\theta)]) d\omega$$

is a strictly increasing function of r_x. We have

$$(6.13) \qquad \frac{dI(r_x)}{dr_x} = \int_{S(d_0)} nd_0 \cos [\gamma(\theta)] \exp (nr_x d_0 \cos [\gamma(\theta)]) d\omega.$$

Denote by ω_1 the subset of $S(d_0)$ in which $0 \leq \gamma(\theta) \leq \frac{\pi}{2}$, and by ω_2 the subset in which $\frac{\pi}{2} \leq \gamma(\theta) \leq \pi$. Because of the symmetry of the sphere we have

$$\begin{aligned} &\int_{\omega_2} nd_0 \cos [\gamma(\theta)] \exp (nr_x d_0 \cos [\gamma(\theta)]) \, d\omega \\ &\qquad = \int_{\omega_1} nd_0 \cos [\pi - \gamma(\theta)] \exp (nr_x d_0 \cos [\pi - \gamma(\theta)]) \, d\omega \\ &\qquad = -\int_{\omega_1} nd_0 \cos [\gamma(\theta)] \exp (-nr_x d_0 \cos [\gamma(\theta)]) \, d\omega. \end{aligned}$$

Hence

$$(6.14) \qquad \begin{aligned} \frac{dI(r_x)}{dr_x} = nd_0 \int_{\omega_1} &\cos [\gamma(\theta)] \\ &\{\exp (nd_0 r_x \cos [\gamma(\theta)]) - \exp (-nd_0 r_x \cos [\gamma(\theta)])\} \, d\omega \end{aligned}$$

The right hand side of (6.14) is positive Hence, we have proved that expression (6 11) (or (6 10)) is a strictly increasing function of r_x.

To show that $\beta(\theta)$ is constant on $S(d)$ and is monotonically decreasing with increasing d, let $y_1, \cdots, y_k$ be an orthogonal linear transformation of $x_1, \cdots, x_k$ so that $E(y_1) = \sqrt{\theta_1^2 + \cdots + \theta_k^2}$, $E(y_i) = 0$ $(i = 2, \cdots, k)$. Since $\bar{y}_1^2 + \cdots + \bar{y}_k^2 = \bar{x}_1^2 + \cdots + \bar{x}_k^2$ and since (6.11) depends only on $\bar{x}_1^2 + \cdots + \bar{x}_k^2$, it is seen that the sequence of expression (6.11) formed for any sequence of integers n has a joint distribution which depends only on $\sqrt{\theta_1^2 + \cdots + \theta_k^2}$. Hence $\beta(\theta)$ is constant on any sphere with center at the origin. Since (6.11) is a strictly increasing function of r_x, it can be shown that $\beta(\theta)$ is a monotonically decreasing function of $\sqrt{\theta_1^2 + \cdots + \theta_k^2}$. Hence, we can test the hypothesis H_0 by the sequential process based on (6.10)

If $k = 1$—that is, if we test the mean value of a single normal variate—the

sphere $S(d)$ is a 0-dimensional sphere consisting of the two points $\theta_1 = +d$ and $\theta_1 = -d$ and expression (6.10) reduces to

$$\frac{\frac{1}{2}\frac{1}{(2\pi)^{n/2}}\{\exp[-\frac{1}{2}\Sigma_\alpha(x_\alpha - d_0)^2] + \exp[-\frac{1}{2}\Sigma_\alpha(x_\alpha + d_0)^2]\}}{\frac{1}{(2\pi)^{n/2}}\exp[-\frac{1}{2}\Sigma x_\alpha^2]} \tag{6.15}$$
$$= \tfrac{1}{2}\exp[-\tfrac{1}{2}nd_0^2]\{\exp[n\bar{x}d_0] + \exp[-n\bar{x}d_0]\}.$$

6.2. *Sequential test of a composite hypothesis.* We shall give only a brief outline of the principles on which a sequential test of a composite hypothesis can be based, since they are analogous to those for a simple hypothesis. Let $X_1, \cdots, X_p$ be a set of p random variables and let $f(x_1, \cdots, x_p, \theta_1, \cdots, \theta_k)$ be the joint probability density function of these variables involving k unknown parameters $\theta_1, \cdots, \theta_k$. Denote the set of all possible parameter points $\theta = (\theta_1, \cdots, \theta_k)$ by Ω. Suppose that we wish to test the hypothesis H_0 that the true parameter point θ is contained in the subset ω of Ω. Let $\bar{\omega}$ be the set of all points of Ω which are not contained in ω. Furthermore, let $w_0(\theta)$ and $w_1(\theta)$ be two non-negative functions of θ, called weight functions, such that

$$\int_\omega w_0(\theta)d\theta = 1 \text{ and } \int_{\bar{\omega}} w_1(\theta)d\theta = 1. \tag{6.16}$$

If ω is a surface in the space Ω then the integral over ω is meant to be the surface integral over ω.

In testing a composite hypothesis the probability of an error of the first kind need not necessarily be the same for all points θ in ω. It will, in general, be a function $\alpha(\theta)$ of the true point θ in ω. Similarly the probability of an error of the second kind is a function $\beta(\theta)$ of θ defined for all points in $\bar{\omega}$. Suppose that we wish to construct a sequential test such that the weighted average $\int_\omega w(\theta)\alpha(\theta)\, d\theta$ of the probabilities of errors of the first kind is a given value α, and the weighted average $\int_{\bar{\omega}} w(\theta)\beta(\theta)\, d\theta$ of the probabilities of errors of the second kind is a given value β. Then the following sequential test can be used: Denote by H_0^* the hypothesis that the probability density in the sample space of n observations on $X_1, \cdots, X_p$ is given by

$$p_{0n} = \int_\omega w_0(\theta)[\prod_\alpha f(x_{1\alpha}, \cdots, x_{p\alpha}, \theta_1, \cdots, \theta_k)]\, d\theta \tag{6.17}$$

and by H_1^* the hypothesis that the density in the sample space is given by

$$p_{1n} = \int_{\bar{\omega}} w_1(\theta)[\prod_\alpha f(x_{1\alpha}, \cdots, x_{p\alpha}, \theta_1, \cdots, \theta_k)]\, d\theta. \tag{6.18}$$

The sequential probability ratio test for testing H_0^* against the single alternative H_1^* provides a solution of our problem. If the constants A and B in this sequen-

tial test are chosen so that the probability is α that we reject H_0^* when H_0^* is true, and the probability is β that we accept H_0^* when H_1^* is true, then for this sequential test we have

$$\int_\omega w_0(\theta)\alpha(\theta)\, d\theta = \alpha$$

and

$$\int_{\bar{\omega}} w_1(\theta)\beta(\theta)\, d\theta = \beta.$$

This can be proved in the same way as the corresponding statement in the case of a simple hypothesis.

Frequently we may require a sequential test procedure such that the least upper bound of $\alpha(\theta)$ in ω is equal to a given α and $\beta(\theta)$ is less than or equal to a given β for all points θ whose "distance" (defined in some sense) from ω is greater than or equal to a given positive value d_0. The "distance" of a parameter point θ from ω may be defined by some function $\delta(\theta, \omega)$ which is positive if θ is not in ω and is zero if θ is in ω. The distance function will be chosen in general according to practical needs and mathematical convenience. For reasons similar to those discussed in the case of a simple hypothesis, an appropriate sequential test procedure with the desired properties can be found as follows: Let $\bar{\omega}(d)$ be the set of all points θ for which $\delta(\theta, \omega) \geq d$. Let, furthermore, $w_0(\theta)$ and $w_1(\theta)$ be two weight functions such that

$$\int_\omega w_0(\theta)\, d\theta = \int_{\bar{\omega}(d_0)} w_1(\theta)\, d\theta = 1. \tag{6.19}$$

Denote by H_0^* the hypothesis that the probability density in the sample space of n observations on $X_1, \cdots, X_p$ is given by

$$p_{0n} = \int_\omega w_0(\theta)\left[\prod_{\alpha=1}^{n} f(x_{1\alpha}, \cdots, x_{p\alpha}, \theta)\right] d\theta \qquad (n = 1, 2, \cdots) \tag{6.20}$$

and by H_1^* the hypothesis that the probability density in the sample space of n observations on $X_1, \cdots, X_p$ is given by

$$p_{1n} = \int_{\bar{\omega}(d_0)} w_1(\theta)\left[\prod_{\alpha=1}^{n} f(x_{1\alpha}, \cdots, x_{p\alpha}, \theta)\right] d\theta. \quad (n = 1, 2, \cdots) \tag{6.21}$$

Consider the sequential probability ratio test for testing the simple hypothesis H_0^* against the single alternative H_1^*. For any θ in ω let $\alpha(\theta)$ be the probability of accepting H_1^* when θ is true, and for any θ in $\bar{\omega}$ let $\beta(\theta)$ be the probability of accepting H_0^* when θ is true. It is clear that $\alpha(\theta)$ and $\beta(\theta)$ depend on the constants A and B used in the sequential process and on the weight functions $w_0(\theta)$ and $w_1(\theta)$. For given A, B, $w_0(\theta)$ and $w_1(\theta)$ let $\beta[A, B, w_0(\theta), w_1(\theta)]$ be the least upper bound of $\beta(\theta)$ in $\bar{\omega}(d_0)$ and let $\alpha[A, B, w_0(\theta), w_1(\theta)]$ be the least upper bound of $\alpha(\theta)$ in ω. Consider the difference

$$\Delta\alpha[A, B, w_0(\theta), w_1(\theta)] = \alpha[A, B, w_0(\theta), w_1(\theta)] - \int_\omega w_0(\theta)\alpha(\theta)\, d\theta$$

and

$$\Delta\beta[A, B, w_0(\theta), w_1(\theta)] = \beta[A, B, w_0(\theta), w_1(\theta)] - \int_{\bar{\omega}(d_0)} w_1(\theta)\beta(\theta)\, d\theta.$$

Determine $w_0(\theta)$ and $w_1(\theta)$ so that Max $[\Delta\alpha, \Delta\beta]$ is a minimum. For these weight functions the least upper bound of $\alpha(\theta)$ in ω and the least upper bound of $\beta(\theta)$ in $\bar{\omega}(d_0)$ will be functions of A and B only. Finally, we determine A and B so that the least upper bound of $\alpha(\theta)$ in ω becomes α, and the least upper bound of $\beta(\theta)$ in $\bar{\omega}(d_0)$ becomes β.

The determination of $w_0(\theta)$ and $w_1(\theta)$ involves the solution of problems in the calculus of variations. However, in some important cases the solution of the problem can easily be derived, since weight functions $w_0(\theta)$ and $w_1(\theta)$ can be found for which $\Delta\alpha = \Delta\beta = 0$. Such a situation is given, for instance, in the following case: Let $S(d)$ be the set of all points θ for which $\delta(\theta, \omega) = d$. Suppose that we can find two weight functions $v_0(\theta)$ and $v_1(\theta)$ such that $\int_\omega v_0(\theta)\, d\theta = \int_{S(d_0)} v_1(\theta)\, dS = 1$ (dS denotes the infinitesimal surface element of $S(d_0)$) and the sequential probability ratio test based on

$$\frac{\int_{S(d_0)} v_1(\theta)[\prod_\alpha f(x_{1\alpha}, \cdots, x_{p\alpha}, \theta)]\, dS}{\int_\omega v_0(\theta)[\prod_\alpha f(x_{1\alpha}, \cdots, x_{p\alpha}, \theta)]\, d\theta}$$

has the following properties: (1) $\alpha(\theta)$ is constant in ω, (2) $\beta(\theta)$ is constant on $S(d)$ for any $d \geq d_0$; (3) $\beta(\theta)$ is strictly decreasing with increasing d in the domain $d \geq d_0$. Then for these weight functions we evidently have $\Delta\alpha = \Delta\beta = 0$.

Let us illustrate this by a simple example. Let X be a normally distributed variate with unknown mean μ and unknown variance σ^2. Suppose that we want to test the hypothesis H_0 that $\mu = 0$ and that the distance of the point (μ, σ) from the set ω is defined by $\left|\frac{\mu}{\sigma}\right|$.

The set $S(d_0)$ then consists of all points (μ, σ) for which $\mu = +d_0\sigma$ or $\mu = -d_0\sigma$. The set ω consists of all points $(0, \sigma)$ where σ can take any arbitrary positive value. Let r be a positive value. We define the weight functions $v_{0r}(\sigma)$ and $v_{1r}(\sigma)$ as follows: $v_{0r}(\sigma) = \frac{1}{r}$ if $0 \leq \sigma \leq r$ and equals zero for all other values of σ. The weight function $v_{1r}(\sigma)$ is equal to $\frac{1}{2r}$ if $0 \leq \sigma \leq r$ and $\mu = \pm d_0\sigma$ and equal to zero otherwise.

Then

$$
\begin{aligned}
p_{1n} &= \int_{S(d_0)} v_{1r}(\sigma) \frac{1}{(2\pi)^{n/2}\sigma^n} \exp\left[-\frac{1}{2}\frac{\Sigma(x_\alpha - \mu)^2}{\sigma^2}\right] d\sigma \\
&= \frac{1}{(2\pi)^{n/2}} \frac{1}{2r} \int_0^r \left\{\frac{1}{\sigma^n} \exp\left[-\frac{1}{2}\frac{\Sigma(x_\alpha - d_0\sigma)^2}{\sigma^2}\right]\right. \\
&\qquad \left. + \frac{1}{\sigma^n} \exp\left[-\frac{1}{2}\frac{\Sigma(x_\alpha + d_0\sigma)^2}{\sigma^2}\right]\right\} d\sigma
\end{aligned} \tag{6.22}
$$

and

$$
p_{0n} = \frac{1}{(2\pi)^{n/2}} \frac{1}{r} \left\{\int_0^r \frac{1}{\sigma^n} \exp\left[-\frac{1}{2}\frac{\Sigma x_\alpha^2}{\sigma^2}\right] d\sigma\right\}. \tag{6.23}
$$

Hence

$$
\begin{aligned}
\frac{p_{1n}}{p_{0n}} &= \frac{\frac{1}{2}\int_0^r \frac{1}{\sigma^n} \exp\left[-\frac{1}{2}\frac{\Sigma(x_\alpha - d_0\sigma)^2}{\sigma^2}\right] d\sigma}{\int_0^r \frac{1}{\sigma^n} \exp\left[-\frac{1}{2}\frac{\Sigma x_\alpha^2}{\sigma^2}\right] d\sigma} \\
&\qquad + \frac{\frac{1}{2}\int_0^r \frac{1}{\sigma^n} \exp\left[-\frac{1}{2}\frac{\Sigma(x_\alpha + d_0\sigma)^2}{\sigma^2}\right] d\sigma}{\int_0^r \frac{1}{\sigma^n} \exp\left[-\frac{1}{2}\frac{\Sigma x_\alpha^2}{\sigma^2}\right] d\sigma}.
\end{aligned} \tag{6.24}
$$

We consider the limiting case when $r \to \infty$. Then

$$
\begin{aligned}
\frac{p_{1n}}{p_{0n}} &= \frac{\frac{1}{2}\int_0^\infty \frac{1}{\sigma^n} \exp\left[-\frac{1}{2}\frac{\Sigma(x_\alpha - d_0\sigma)^2}{\sigma^2}\right] d\sigma}{\int_0^\infty \frac{1}{\sigma^n} \exp\left[-\frac{1}{2}\frac{\Sigma x_\alpha^2}{\sigma^2}\right] d\sigma} \\
&\qquad + \frac{\frac{1}{2}\int_0^\infty \frac{1}{\sigma^n} \exp\left[-\frac{1}{2}\frac{\Sigma(x_\alpha + d_0\sigma)^2}{\sigma^2}\right] d\sigma}{\int_0^\infty \frac{1}{\sigma^n} \exp\left[-\frac{1}{2}\frac{\Sigma x_\alpha^2}{\sigma^2}\right] d\sigma}.
\end{aligned} \tag{6.25}
$$

The sequential test based on the ratio (6.25) provides a solution of the problem if it can be shown to have the following three properties: (1) $\alpha(\theta)$ is constant in ω; (2) $\beta(\theta)$ is only a function of $\left|\frac{\mu}{\sigma}\right|$; (3) $\beta(\theta)$ is monotonically decreasing with increasing $\left|\frac{\mu}{\sigma}\right|$. Denote $\frac{\sum_{\alpha=1}^n x_\alpha}{n}$ by $\bar{x}$ and $\sum_{\alpha=1}^n (x_\alpha - \bar{x})^2$ by S^2. Since the distribution of $\left|\frac{\bar{x}}{S}\right|$ depends only on $\left|\frac{\mu}{\sigma}\right|$, the first two properties are proved if we

show that the ratio (6 25) is a single valued function of $\left|\frac{\bar{x}}{S}\right|$.

First we show that the numerator of the ratio (6 25) is a homogenous function of $(x_1, \cdots, x_n)$ of degree $-(n-1)$ In fact, making the transformation $\sigma = \lambda t$ we obtain

$$\int_0^\infty \left\{\frac{1}{\sigma^n} \exp\left[-\frac{1}{2}\frac{\Sigma(\lambda x_\alpha - d_0\sigma)^2}{\sigma^2}\right] + \frac{1}{\sigma^n}\exp\left[-\frac{1}{2}\frac{\Sigma(\lambda x_\alpha + d_0\sigma)^2}{\sigma^2}\right]\right\} d\sigma$$

$$= \int_0^\infty \left\{\frac{1}{(\lambda t)^n} \exp\left[-\frac{1}{2}\frac{\Sigma(x_\alpha - d_0 t)^2}{t^2}\right] + \frac{1}{(\lambda t)^n}\exp\left[-\frac{1}{2}\frac{\Sigma(x_\alpha + d_0 t)^2}{t^2}\right]\right\} d(\lambda t)$$

$$= \frac{1}{\lambda^{n-1}}\int_0^\infty \left\{\frac{1}{t^n} \exp\left[-\frac{1}{2}\frac{\Sigma(x_\alpha - d_0 t)^2}{t^2}\right] + \frac{1}{t^n}\exp\left[-\frac{1}{2}\frac{\Sigma(x_\alpha + d_0 t)^2}{t^2}\right]\right\} dt.$$

This proves that the numerator of (6.25) is a homogenous function of $-(n-1)$ degree. Similarly, it can be shown that the denominator of (6.25) is also a homogenous function of degree $-(n-1)$. Thus the ratio (6.25) is a homogenous function of zero degree in the variables $x_1, \cdots, x_n$.

It can be seen that (6.25) is a function of the two expressions Σx_α^2 and Σx_α only; i.e.,

$$\frac{p_{1n}}{p_{0n}} = \phi(\Sigma x_\alpha^2, \Sigma x_\alpha). \tag{6.26}$$

Let $v = \left|\sqrt{\Sigma x_\alpha^2}\right|$. Since (6.26) is a homogenous function of zero degree, its value is not changed by substituting $\frac{x_\alpha}{v}$ for x_α. Hence,

$$\frac{p_{1n}}{p_{0n}} = \phi\left[\sum_\alpha\left(\frac{x_\alpha}{v}\right)^2, \sum\frac{x_\alpha}{v}\right] = \phi\left[1, \frac{n\bar{x}}{v}\right]. \tag{6.27}$$

Since $\phi(\Sigma x_\alpha^2, -\Sigma x_\alpha) = \phi(\Sigma x_\alpha^2, \Sigma x_\alpha)$, we see that

$$\frac{p_{1n}}{p_{0n}} = \psi\left[\frac{\bar{x}^2}{v^2}\right].$$

Since $\frac{(\bar{x})^2}{v^2}$ is a single valued function of $\left|\frac{\bar{x}}{S}\right|$, we have proved that $\frac{p_{1n}}{p_{0n}}$ is a single valued function of $\left|\frac{\bar{x}}{S}\right|$.

In order to prove property (3) of the sequential test based on the ratio (6.25), we have merely to show that (6.25) is a strictly increasing function of $\left|\frac{\bar{x}}{S}\right|$. Since $\frac{\bar{x}^2}{v^2}$ is a strictly increasing function of $\left|\frac{\bar{x}}{S}\right|$, we have only to show that (6.25) is a strictly increasing function of $\frac{\bar{x}^2}{v^2}$. The latter statement is obviously proved if we show that (6.25) increases with increasing value $|\bar{x}|$ while keeping

v fixed. For fixed value of v the denominator of (6.25) is constant. Thus, we have merely to show that the numerator of (6.25) increases with increasing $|\bar{x}|$ while keeping v fixed. This follows easily from the fact that

$$\exp\left[\frac{(\Sigma x_\alpha)\, d_0}{\sigma}\right] + \exp\left[\frac{-(\Sigma x_\alpha)\, d_0}{\sigma}\right]$$

is a strictly increasing function of $|\bar{x}|$.

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NON-PARAMETRIC ESTIMATION. I. VALIDATION OF ORDER STATISTICS

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1. Summary. Previous work on non-parametric estimation has concerned three problems: (*i*) confidence intervals for an unknown quantile, (*ii*) population tolerance limits, (*iii*) confidence bands for an unknown cumulative distribution function (*cdf*). For problem (*iii*) a solution has been available which is valid for any *cdf* whatever, but for (*i*) and (*ii*) it has heretofore been assumed that the population has a continuous probability density This paper validates the existing solutions of (*i*) and (*ii*) assuming only a continuous *cdf*. It then modifies these solutions so that they are valid for any *cdf* whatever.

2. Introduction. There are three problems of non-parametric estimation (we exclude point-estimation) for which fairly satisfactory solutions are available; their present status was summarized in a recent paper [4]. The purpose of this series of articles is to extend and complete the theory of non-parametric estimation in directions of both theoretical and practical interest

In this series we shall employ the following conventions of notation: We distinguish between a random variable and an arbitrary point in the Euclidean space containing its domain by using a capital Roman letter for the former and the corresponding lower case Roman letter for the latter. Thus if X is a (scalar) random variable, and x a real number or $\pm\infty$, we speak of the probability that $X \leq x$ and denote it by $Pr\{X \leq x\}$. Roman capitals will also be used to denote cumulative distribution functions[1] (*cdf*'s): A monotone non-decreasing function $F(x)$ will be called the *cdf* of X if $F(x + 0) = Pr\{X \leq x\}$ The definition of $F(x)$ at its points of discontinuity will be immaterial. Again, $E = (X_1, \cdots, X_n)$ will denote a random sample from a population with *cdf* $F(x)$, whereas $e = (x_1, \cdots, x_n)$ will denote a point in the sample space R_n. If t is a function of e only, $t = \varphi(e)$, then the random variable $T = \varphi(E)$ is a statistic. The *order statistics* of the sample E are defined to be $-\infty, Z_1, \cdots, Z_n, +\infty$, where $z_1 \leq z_2 \leq \cdots \leq z_n$ is a rearrangement of $x_1, x_2, \cdots, x_n$. We shall write $Z_0 = -\infty, Z_{n+1} = +\infty$. The device of including $+\infty$ and $-\infty$ among the order statistics will enable us to avoid special statements to cover the case of one-sided estimation. Confidence coefficients will be denoted by $1 - \alpha$. Finally, it will be convenient to symbolize[2] the following three classes of *cdf*'s: Ω_0 is the class of all univariate *cdf*'s F, Ω_2, the class of all continuous F; Ω_4, the class of all F with continuous derivative $F'(x)$.

[1] One of the authors wishes to point out the need of a clear, concise, and adequate term for this basic and important concept

[2] The notation follows [3]

We now list the three problems. In each case it is understood that the solution sought is to be valid for all *cdf*'s in some chosen class. The names[3] associated with the problems are (*i*) W. R. Thompson, K. R. Nair, (*ii*) Wilks, (*iii*) Wald, Wolfowitz, Kolmogoroff.

(*i*) To find confidence intervals for an unknown quantile q_p, where q_p is defined by $F(q_p) = p$, $0 < p < 1$; in other words, to find statistics T_1, T_2 such that[4]

$$Pr\{T_1 \leq q_p \leq T_2 \mid F\} = 1 - \alpha. \tag{1.1}$$

(*ii*) To find tolerance limits T_1, T_2 which, with confidence $1 - \alpha$, will cover a proportion b or more of the population, that is,

$$Pr\{F(T_2) - F(T_1) \geq b \mid F\} = 1 - \alpha. \tag{1.2}$$

(*iii*) To find a confidence band for an unknown *cdf* F, that is, a random region $R(E)$ in the x,y-plane such that

$$Pr\{R(E) \text{ covers } g \mid F\} = 1 - \alpha, \tag{1.3}$$

where g is the graph of $y = F(x)$.

The existing solutions of problem (*iii*) are known to be valid for F in Ω_2, but those of problems (*i*) and (*ii*) have been validated only for F in Ω_4. The extension to F in Ω_2 is an immediate consequence of the theorem in section 4; this section also contains a discussion of some other implications of the theorem. In section 5 the appropriate modifications of the solutions of problems (*i*) and (*ii*) are found which extend their validity to the general case F in Ω_0. Whereas Pitman ([1]; also [4], p. 310) has shown how non-parametric tests may be extended to the possibly discontinuous case, the only solution of the three estimation problems previously extended to this case is that of Kolmogoroff for problem (*iii*). Extension from Ω_2 to Ω_0 is of considerable practical interest, not only in the case of populations ordinarily considered discrete, but also as affecting the problem of the finiteness of the number of significant figures in measurements and the resulting occurrence of "ties" in ranked measurements. Before making these extensions we discuss in the next section the transformations on which they are based.

3. Two useful transformations of random variables. We shall reserve the symbol X^* for a random variable having a uniform distribution on the interval from 0 to 1. Its *cdf* is

$$U(x^*) = Pr\{X^* \leq x^*\} = \begin{cases} 0 \text{ if } x^* < 0, \\ x^* \text{ if } 0 \leq x^* \leq 1, \\ 1 \text{ if } x^* > 1. \end{cases} \tag{1.4}$$

[3] For bibliography see [4].

[4] The notation $Pr\{R \mid F_0\}$ denotes the probability of the relation R being true, calculated under the assumption that the *cdf* of the population is $F_0(x)$

The device of transforming from any random variable X with *cdf* F in Ω_2 to one with *cdf* U was early used by Karl Pearson and more recently by many others; it is known in the literature as the "probability integral transformation." We define the transformation $x^* = h_F(x)$ as follows: For $-\infty < x < +\infty$, $h_F(x) = F(x)$, $h_F(+\infty) = +\infty$, $h_F(-\infty) = -\infty$ If F is in Ω_2, the following statements are evident for the transform $X^* = h_F(X)$: X^* has $U(x^*)$ as its *cdf*. With $X_i^* = h_F(X_i)$, a random sample $E = (X_1, \cdots, X_n)$ from F transforms into a random sample $E^* = (X_1^*, \cdots, X_n^*)$ from U. The order statistics $\{Z_i\}$ of E transform into the order statistics $\{Z_i^*\}$ of E^* with $Z_i^* = h_F(Z_i)$, $i = 0, 1, \cdots, n+1$.

It is easily seen that if F is not in Ω_2, the above transformation $Y = h_f(X)$ does not give Y the *cdf* U; indeed, if F is not in Ω_2, the *cdf* of any single-valued function Y of X is also not in Ω_2, for there will be at least one point $x = x_0$ with positive probability, and likewise for its transform y_0. Nevertheless our arguments in section 4 depend on relating a random variable with arbitrary *cdf* F in Ω_0 to the uniformly distributed X^*. While it is not possible to transform from X to X^*, without introducing a further random process, *it is possible to transform directly from X^* to X.* This suffices for our needs We shall always denote this transformation by $X = g_F(X^*)$. The following definition of the function $x = g_F(x^*)$ makes it independent of the normalization of F at its discontinuities:

$$F(x - 0) \leq U(x^*) \leq F(x + 0) \tag{1.5}$$

A sketched diagram may aid the reader in following the argument: To every x^* $(-\infty \leq x^* \leq +\infty)$ there corresponds at least one x, and this x is unique unless it lies in an interval to which F assigns zero probability In the latter case we shall assume that some x in the interval is designated to be $g_F(x^*)$. It will be seen that it is immaterial which x is thus chosen. However if $x = -\infty$ or $+\infty$ is in an interval of constancy of F we specify $g_F(-\infty) = -\infty$, $g_F(+\infty) = +\infty$.

To prove that $g_F(X^*)$ has the *cdf* $F(x)$ and thus can be identified with X, it is sufficient to prove that $Pr\{g_F(X^*) \leq x\} = F(x + 0)$. Now $g_F(X^*) \leq x$ if and only if $X^* \leq x_+^*$, where

$$x_+^* = \sup_{x = g_F(x^*)} x^*.$$

Hence $Pr\{g_F(X^*) \leq x\} = Pr\{X^* \leq x_+^*\} = U(x_+^*) = F(x + 0)$. It follows that a random sample E^* from U transforms into a random sample E from F. The transformation preserves the relation "$\leq$," that is, if $x_a = g_F(x_a^*)$, $x_b = g_F(x_b^*)$, then $x_a^* \leq x_b^*$ implies $x_a \leq x_b$. This means that the order statistics $\{Z_i^*\}$ of E^* transform into the order statistics $\{Z_i\}$ of E. We remark that $x_a^* < x_b^*$ does not imply $x_a < x_b$, there is trouble when $x_b^* \leq 0$ or $x_a^* \geq 1$, and more serious trouble if x_a^* and x_b^* both go into the same discontinuity of F. However, we shall need to utilize the fact that $x_a < x_b$ implies $x_a^* \leq x_b^*$.

4. Extension to continuous cdf's. A sufficient condition on T_1 and T_2 for a solution (1.2) of problem (*ii*) to be valid for all F in Ω_2 is clearly that the joint distribution of $F(T_1)$ and $F(T_2)$ be independent of F in Ω_2. If $Pr\{F(T_i) = p \mid F\} = 0$ $(i = 1, 2)$, then (1.1) is equivalent to

$$Pr\{F(T_1) \leq p \leq F(T_2) \mid F\} = 1 - \alpha, \tag{1.6}$$

and so a sufficient condition that a solution (1.1) of problem (*i*) be valid for all F in Ω_2 is again that the joint distribution of $F(T_1)$ and $F(T_2)$ be independent of F in Ω_2. We are thus led to consider sufficient conditions on a set $T_1, T_2, \cdots, T_r$ of statistics, which will insure that the joint distribution of $F(T_1)$, $F(T_2)$, $\cdots$, $F(T_r)$ be independent of F in Ω_2.

THEOREM: *A sufficient condition for the joint distribution of $F(T_1)$, $F(T_2)$, $\cdots$, $F(T_r)$ to be independent of F in Ω_2 is that the $\{T_j\}$ be a subset of the order statistics $\{Z_i\}$ of the sample.*

To prove the theorem it will suffice to show that the joint distribution of the set of n random variables $F(Z_1)$, $F(Z_2)$, $\cdots$, $F(Z_n)$ is independent of F in Ω_2. Let the *cdf* of the joint distribution be

$$G_F(\lambda_1, \lambda_2, \cdots, \lambda_n) = Pr\{F(Z_1) \leq \lambda_1, \cdots, F(Z_n) \leq \lambda_n \mid F\}. \tag{1.7}$$

Employing the transformation $x^* = h_F(x)$ discussed in section 3, we see that the above probability equals

$$Pr\{Z_1^* \leq \lambda_1, \cdots, Z_n^* \leq \lambda_n\}, \tag{1.8}$$

where $Z_0^*, Z_1^*, \cdots, Z_{n+1}^*$ are the order statistics of a random sample E^* from the uniform *cdf* U. But this probability does not depend on F.

Since the existing solutions of problems (*i*) and (*ii*) are obtained by taking T_1 and T_2 to be order statistics, we have validated these solutions for all F in Ω_2. That the existing solutions of problem (*iii*) are valid for F in Ω_2 has been demonstrated by their authors; this is however also an easy consequence of the above theorem. The sufficiency condition expressed by this theorem together with a necessity condition of Robbins' [2] may indicate a natural path to the formulation and solution of further problems of non-parametric estimation.

From a theoretical point of view it is of interest to note that even in those pathological cases where no probability density function exists for the *cdf* F in Ω_2 (F is non-absolutely continuous), the joint distribution (1.7) of $F(Z_1)$, $F(Z_2)$, $\cdots$, $F(Z_n)$ always possesses a density. That this density is $n!$ for $0 \leq F(Z_1) \leq F(Z_2) \leq \cdots \leq F(Z_n) \leq 1$, and zero elsewhere, is evident if we consider (1.8). By "integrating out" the other variables we are led to the following practically useful result (it is well known for F in Ω_4): Choose any set $\{r_j\}$ of s integers $(1 \leq r_1 < r_2 < \cdots < r_s \leq n)$, and consider the joint distribution of $F(Z_{r_1})$, $F(Z_{r_2})$, $\cdots$, $F(Z_{r_s})$. This has a probability density function $f(t_1, t_2, \cdots, t_s)$, providing F is in Ω_2, given by the formula

$$(1.9) \qquad f(t_1, t_2, \cdots, t_s) = \frac{n!\, t_1^{r_1-1}(1 - t_s)^{n-r_s}}{(r_1 - 1)!\,(n - r_s)!} \prod_{i=1}^{s-1} \frac{(t_{i+1} - t_i)^{r_{i+1}-r_i-1}}{(r_{i+1} - r_i - 1)!}$$

for $0 \leq t_1 \leq t_2 \leq \cdots \leq t_s \leq 1$, and $f = 0$ elsewhere. As is conventional, the result of applying $\prod_{i=1}^{0}$ is to be interpreted as unity, and the meaning of f is given by

$$Pr\{F(Z_{r_i}) \leq a_i (i = 1, 2, \cdots, s)|F\}$$
$$= \int_{-\infty}^{a_1} \int_{-\infty}^{a_2} \cdots \int_{-\infty}^{a_s} f(t_1, t_2, \cdots, t_s)\, dt_s \cdots dt_2\, dt_1.$$

5. Extension to discontinuous cdf's. Suppose we have a solution of problem (i) based on order statistics and hence valid for F in Ω_2, say,

$$(1.10) \qquad Pr\{Z_k \leq q_p \leq Z_t \mid F\} = 1 - \alpha,$$

where $0 \leq k < t \leq n + 1$ In particular this is valid for the uniform case,

$$(1.11) \qquad Pr\{Z_k^* \leq p \leq Z_t^*\} = 1 - \alpha.$$

We now transform from the uniform *cdf* U to an arbitrary F in Ω_0 by means of the transformation $x = g_F(x^*)$ described in section 3. Suppose q_p is defined by $q_p = g_F(p)$. This means the quantile q_p of the distribution with *cdf* F is determined from the relation

$$F(q_p - 0) \leq p \leq F(q_p + 0),$$

which assigns to the quantile its usual meaning if $F(x)$ is continuous and non-constant at $x = q_p$, and a sensible definition if F is discontinuous or constant at q_p. From the discussion in section 3 we have

$$(Z_k < q_p < Z_t) \text{ implies } (Z_k^* \leq p \leq Z_t^*) \text{ implies } (Z_k \leq q_p \leq Z_t),$$

and hence the probability relations

$$Pr\{Z_k < q_p < Z_t \mid F\} \leq Pr\{Z_k^* \leq p \leq Z_t^*\} \leq Pr\{Z_k \leq q_p \leq Z_t \mid F\}.$$

Substituting (1.11), we have

$$(1.12) \quad Pr\{Z_k < q_p < Z_t \mid F\} \leq 1 - \alpha \leq Pr\{Z_k \leq q_p \leq Z_t \mid F\}.$$

The statistical interpretation of (1.12) is the following: Consider any solution (1.10) of problem (i), giving a confidence interval for the quantile q_p, valid for F in Ω_2. Then with the same values of n, k, t, and α, the probability of the random interval from Z_k to Z_t covering the unknown quantile q_p is $\leq 1 - \alpha$ for the open interval, $\geq 1 - \alpha$ for the closed interval, no matter what the unknown *cdf* F. If F is continuous, the two probabilities are of course equal.

To extend the solution of problem (ii) to the general case F in Ω_0, suppose we have a solution (1 2) using order statistics, say $T_1 = Z_k$, $T_2 = Z_t$ $(0 \leq k < t \leq n + 1)$. Such a solution will be valid for all F in Ω_2, in particular for $F = U$,

$$Pr\{U(Z_t^*) - U(Z_k^*) \geq b\} = 1 - \alpha.$$

Given now any arbitrary distribution F, we again use the transformation $x = g_F(x^*)$. From (1 5),

$$F(Z_i - 0) \leq U(Z_i^*) \leq F(Z_i + 0) \qquad (i = k, t).$$

Hence

$$B_- \leq B^* \leq B_+,$$

where

$$B_- = F(Z_t - 0) - F(Z_k + 0),$$
$$B^* = U(Z_t^*) - U(Z_k^*),$$
$$B_+ = F(Z_t + 0) - F(Z_k - 0).$$

The implications

$$(B_- \geq b) \text{ implies } (B^* \geq b) \text{ implies } (B_+ \geq b)$$

yield the relations

$$Pr\{B_- \geq b\} \leq Pr\{B^* \geq b\} \leq Pr\{B_+ \geq b\}.$$

These may be written

$$(1.13) \quad Pr\{F(Z_t - 0) - F(Z_k + 0) \geq b \mid F\} \leq 1 - \alpha$$
$$\leq Pr\{F(Z_t + 0) - F(Z_k - 0) \geq b \mid F\}$$

To interpret (1.13), let us say that a Borel set S covers a proportion π of a population with *cdf* $F(x)$ if $\int_S dF(x) = \pi$. If S is an interval from x' to x'', then the proportion covered by S is $F(x'' + 0) - F(x' - 0)$ if S is closed, and $F(x'' - 0) - F(x' + 0)$ if S is open. The proportion covered by a point x_0 is the jump $F(x_0 + 0) - F(x_0 - 0)$ of the *cdf* F at x_0. The statistical meaning of (1 13) is now clear: For the random interval from Z_k to Z_t, the probability that the open interval cover a proportion $\geq b$ of the population is $\leq 1 - \alpha$, the probability that the closed interval cover a proportion $\geq b$ of the population is $\geq 1 - \alpha$, regardless of the population. Again, for a continuous F the two probabilities are equal.

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ON A TEST FOR RANDOMNESS BASED ON SIGNS OF DIFFERENCES[1]

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1. Introduction. It has been pointed out by J. Wolfowitz [1] that we cannot expect a test for randomness to be most powerful with respect to every possible alternative. It is therefore necessary to find tests designed to distinguish a random sample of observations from the same population from a sample coming from some particular class Ω of distributions. Such a test need be consistent in the sense of Wald and Wolfowitz [2] only with respect to alternatives in the class Ω.

Let $x_1, \cdots, x_n$ be the measurable quality characteristics of n units of a manufactured article. We shall assume that the distribution of x_i is continuous. According to Shewhart the production process is termed "under statistical control" if $x_1, \cdots, x_n$ can be regarded as a random sample of n independent items each coming from the same population with known or unknown distribution function.

In a random sample $p_i = p(x_i > x_{i+1}) = \frac{1}{2}$, where $P(E)$ denotes the probability that E will hold. The class Ω of alternatives which we shall consider is described as follows. The cumulative distribution of x_i is f_i and the f_i, $i = 1, 2, \cdots$, are such that

$$p_i = \tfrac{1}{2} + \epsilon_i, \qquad \sum_{i=1}^{i=n} \epsilon_i = \lambda_n(n-1), \qquad \liminf_{n\to\infty} \lambda_n = \lambda > 0.$$

Such a situation may, for instance, obtain of the production process is under statistical control except for occasionally but not too infrequently occurring periods during which the quality of the product decreases, after which decrease statistical control is immediately restored. If the decreases in quality are sharp enough or the periods of decrease long enough, then the alternative will belong to the class Ω described before.

To give a practical example; consider a drill, which after some period of use will wear off so that the quality of the manufactured article will decrease until the drill is exchanged. After replacement of the drill by a new one, statistical control is immediately restored. Now, if the drill is not replaced in time, the periods of decrease in quality will be long and the rate of decrease will become rapid so that the sequence of distribution functions will satisfy the conditions of the class Ω. A similar situation occurs also in time studies. For instance, in the foregoing example, the time necessary for drilling one hole will tend to increase when the drill is too long in use.

The following test first proposed by Moore and Wallis [3] for the study of

[1] Research under a grant of the Research Foundation of the Ohio State University

economic time series seems appropriate for our purpose: Let $x_1, \cdots, x_n$ be the sample and form the sequence $x_2 - x_1, \cdots, x_n - x_{n-1}$. Let S be the number of negative differences in this sequence. Clearly, the distribution of S is independent of the distribution of x_i provided the sample is an independent random sample from a continuous distribution. Under one of the alternatives of the class Ω, S will in a sample of n tend to be larger than in a random sample if $\lambda_n > 0$. Hence S may be used as a statistic to distinguish between randomness and any of the alternatives of the class Ω. The distribution of S was tabulated by Moore and Wallis [3] for $n \leq 12$. They also found empirically that S approaches a normal distribution. The asymptotic normality of the distribution of S can be proved rigorously in a way analogous to the proof of Theorem 1 of a paper by Wolfowitz [4]. The first four moments of S were obtained by Moore and Wallis. The fourth moment, however, only by empirical methods. In this paper we shall derive a formula which makes it possible to compute the moments of S recursively. With the help of this formula we shall indicate an alternative proof of the asymptotic normality of S using the method of moments. Finally, we shall derive a lower bound for the power of the S test with respect to alternatives in Ω valid for large n and depending only on λ_n.

2. The moments of S: Let $P_n(S)$ be the number of permutations in n variables with S negative differences. MacMahon [5] has shown that

$$(1) \qquad P_n(S) = (S + 1)P_{n-1}(S) + (n - S)P_{n-1}(S - 1).$$

Using (1) Moore and Wallis [3] have tabulated $P\left(\left|S - \frac{n-1}{2}\right| \geq \left|\bar{S} - \frac{n-1}{2}\right|\right)$. In using their table for our purpose, one has to keep in mind that we are using a one tail region, therefore $P(S \geq \bar{S})$ is for $S \geq \frac{n-1}{2}$ one half of the value tabulated by Moore and Wallis.

Clearly the first moment of S is $\frac{n-1}{2}$, since the expected value of $-$ signs equals the expected value of $+$ signs. To find higher moments we multiply (1) by $\left(S - \frac{n-1}{2}\right)^i$ divide by $n!$ and sum over S. Then we obtain

$$(2) \qquad E_n\left[\left(S - \frac{n-1}{2}\right)^i\right] = \frac{1}{n} E_{n-1}\left[\left(S - \frac{n-1}{2}\right)^i (S + 1)\right] + \frac{1}{n} E_{n-1}\left[(n - S - 1)\left(S - \frac{n-1}{2} + 1\right)^i\right],$$

where $E_n[f(S)]$ denotes the expectation of $f(S)$ in permutations of n variables. From (2) we have

$$E_n\left[\left(S - \frac{n-1}{2}\right)^i\right] = \frac{1}{n} E_{n-1}\left[\left(S - \frac{n-2}{2} - \frac{1}{2}\right)^i \left(S - \frac{n-2}{2}\right)\right. \\ + \tfrac{1}{2}E_{n-1}\left[\left(S - \frac{n-2}{2} - \frac{1}{2}\right)^i\right] \\ - \frac{1}{n} E_{n-1}\left[\left(S - \frac{n-2}{2} + \frac{1}{2}\right)^i \left(S - \frac{n-2}{2}\right)\right] \\ + \frac{1}{2} E_{n-1}\left[\left(S - \frac{n-2}{2} + \frac{1}{2}\right)^i\right].$$

Putting $S - E(S) = x$ we obtain

$$(3)\qquad E_n(x^i) = \frac{1}{n} E_{n-1}[x(x - \tfrac{1}{2})^i - x(x + \tfrac{1}{2})^i] + \tfrac{1}{2}E_{n-1}[(x + \tfrac{1}{2})^i + (x - \tfrac{1}{2})^i].$$

From the symmetry of the distribution as well as from 3 it may be seen that all odd moments are 0 and therefore

$$\tfrac{1}{2}E[(x + \tfrac{1}{2})^{2i} + (x - \tfrac{1}{2})^{2i}] = E(x + \tfrac{1}{2})^{2i}$$

$$E[x(x - \tfrac{1}{2})^{2i} - x(x + \tfrac{1}{2})^{2i}] = -2E[(x + \tfrac{1}{2})^{2i+1}] + E(x + \tfrac{1}{2})^{2i}.$$

Hence we obtain from 2

$$(4)\qquad E_n(x^{2i+1}) = 0, \qquad i = 0, 1, \cdots$$

$$E_n(x^{2i}) = \frac{n+1}{n} E_{n-1}[(x + \tfrac{1}{2})^{2i}] - \frac{2}{n} E_{n-1}[(x + \tfrac{1}{2})^{2i+1}]$$

If all moments below the $2i$th moment are known (4) becomes a difference equation whose solution yields the $2i$th moment for $n \geq 2i$. Thus one obtains

$$\sigma_n^2(S) = E_n(x^2) = \frac{n+1}{12}, \qquad E_n(x^4) = \frac{5(n+1)^2 - 2(n+1)}{240},$$

$$E_n(x^6) = \frac{35(n+1)^3 - 42(n+1)^2 + 16(n+1)}{4032}.$$

It is not difficult to prove from (4) by induction that $\lim_{n\to\infty} \frac{E_n x^{2i}}{\sigma_n^{2i}(S)} = (2i-1)(2i-3) \cdots 3.1$. To do this one proves first by induction that $E_n(x^{2i})$ is for $n \geq 2i$ a polynomial in n of degree i. It can then be proved by induction that the first coefficient of this polynomial is $(2i - 1)(2i - 3) \cdots 3.1/12^i$ from which the assertion follows. Since $(2i - 1) \cdots 3.1$ are the moments of a normal distribution with variance 1 it follows that $\frac{\left(S - \frac{n-1}{2}\right)\sqrt{12}}{\sqrt{n+1}}$ is in the limit normally distributed with mean 0 and variance 1. This result follows, however, also easily from Theorem 2 of a paper by Wolfowitz [4].

It is also possible to show by induction from equation (4) that for $n \geq 2i$ the $2i$th moment of S is smaller than the corresponding moment of a normal distribution with variance $\frac{n+1}{12}$.

3. The power of the S test. Let us assume now that one of the alternatives of the class Ω is true This is to say $p_i = P(x_i > x_{i+1}) = \frac{1}{2} + \epsilon_i$, $\sum \epsilon_i = \lambda_n(n-1)$, $\liminf \lambda_n = \lambda > 0$. Let

$$z_i = \begin{cases} 1 \text{ if the } i\text{th sign is } -, \\ 0 \text{ if the } i\text{th sign is } +. \end{cases}$$

We shall show that

$$P(z_{i+1} = 1 \mid z_i = 1) \leqq P(z_{i+1} = 1).$$

We have

$$\left[\int_{-\infty}^{x_1} df_2(x_2) \int_{-\infty}^{x_2} df_3(x_3)\right] \int_{x_1}^{\infty} df_2(x_2) \leq \int_{-\infty}^{x_1} df_2(x_2) \int_{x_1}^{\infty} df_2(x_2) \int_{-\infty}^{x_1} df_3(x_3)$$
$$\leq \int_{-\infty}^{x_1} df_2(x_2) \left[\int_{x_1}^{\infty} df_2(x_2) \int_{-\infty}^{x_2} df_3(x_3).\right.$$

Adding $\int_{-\infty}^{x_1} df_2(x_2) \left[\int_{-\infty}^{x_1} df_2(x_2) \int_{-\infty}^{x_2} df_3(x_3)\right]$ to both sides of this inequality we have

$$\int_{-\infty}^{x_1} df_2(x_2) \int_{-\infty}^{x_2} df_3(x_3) \leq \int_{-\infty}^{x_1} df_2(x_2) \left[\int_{-\infty}^{+\infty} df_2(x_2) \int_{-\infty}^{x_2} df_3(x_3).\right.$$

Integrating both sides with respect to x_1, we obtain

$$\int_{-\infty}^{+\infty} df_1(x_1) \int_{-\infty}^{x_1} df_2(x_2) \int_{-\infty}^{x_2} df_3(x_3)$$
$$\leq \left[\int_{-\infty}^{+\infty} df_1(x_1) \int_{-\infty}^{x_1} df_2(x_2)\right]\left[\int_{-\infty}^{\infty} df_2(x_2) \int_{-\infty}^{x_2} df_3(x_3)\right]$$

or

$$P(z_1 = 1 \quad \text{and} \quad z_2 = 1) \leq P(z_1 = 1) \cdot P(z_2 = 1).$$

From this it follows that $\sigma_{z_i z_{i+1}} \leq 0$. Since $\sigma_{z_i}^2 = \frac{1}{4} - \epsilon_i^2$ we have $\sigma_s^2 \leq \frac{n-1}{4} - \sum_{i=1}^{i=n-1} \epsilon_i^2 \leqq \frac{n-1}{4}(1 - 4\lambda_n^2)$. Moreover $E(S) = \frac{n-1}{2} + \lambda_n(n-1)$.

Let $\lambda' = \lambda$ if $\lambda < \frac{1}{2}$ and $0 \leq \lambda' < \lambda$ if $\lambda = \frac{1}{2}$. The critical region is for sufficiently large n given approximately by $S > \frac{n-1}{2} + t\sqrt{\frac{n+1}{12}}$, where t

depends on the level of significance α and must be chosen so that $\frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-\frac{1}{2}x^2}\, dx = \alpha$. Hence, if we can show that under any alternative H of the class Ω and for any $\epsilon > 0$

$$(5) \qquad P\left(S \geq E(S) - \frac{t}{2}\sqrt{(n-1)(1-4\lambda'^2)}\right) \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-t} e^{-\frac{1}{2}x^2}\, dx + \epsilon$$

for every $t \geq \bar{t} > 0$, $n \geq N(\epsilon, H, \bar{t})$, then we shall be able to give a lower bound for the power of the S test. The power of the S test is approximately given by $P\left(S \geq \frac{n-1}{2} + t\sqrt{\frac{n+1}{12}}\right)$.

From (5) we have

$$(6) \qquad P\left(S \geq \frac{n-1}{2} + t\sqrt{\frac{n+1}{12}}\right) \geq \frac{1}{\sqrt{2\pi}} \int_{\frac{t\sqrt{n+1} - 2\lambda_n(n-1)\sqrt{3}}{\sqrt{(3n-3)(1-4\lambda'^2)}}}^{\infty} e^{-\frac{1}{2}x^2}\, dx - \epsilon$$

$$\text{for} \quad \frac{t\sqrt{n+1} - 2\lambda_n(n-1)\sqrt{3}}{\sqrt{3(n-1)(1-4\lambda'^2)}} < -\bar{t} < 0, \qquad n \geq N(\epsilon, H, \bar{t}).$$

The author considers it safe to assume that (6) holds with a fairly small ϵ for $n \geq 12$ if λ' in (6) is replaced by λ'_n where $\lambda'_n = \lambda_n$ if $\lambda_n < \frac{1}{2}$ and $\lambda'_n < \frac{1}{2}$ if $\lambda_n = \frac{1}{2}$ and if λ'_n is not too close to $\frac{1}{2}$. He bases this belief on the rapidity with which the distribution of S approaches normality under the null hypothesis of randomness, and on the fact that at least under the 0 hypothesis the moments of S are smaller than the corresponding moments of a normal distribution. It may also be seen from the following derivation of (6) that in many cases the power of the S test will be considerably above the lower bound given in (6).

To prove (5), we need the following two lemmas

Lemma 1. *Let $P(x \leqq t) = f(t)$. Let further $E(z) = 0$, $E(z^2) = \epsilon$. Then for every $\delta > 0$*

$$(7) \qquad f(t+\delta) + \frac{\epsilon}{\delta^2} \geq P(x + z \leq t) \geq f(t - \delta) - \frac{\epsilon}{\delta^2}.$$

Proof: Applying Tschebycheff's inequality we have

$$P(x + z \leq t) \leq P(x \leq t + \delta) + P(x \geq t + \delta \text{ and } z \leq -\delta)$$

$$\leq P(x \leq t + \delta) + P(z \leq -\delta) \leq f(t + \delta) + \frac{\epsilon}{\delta^2},$$

$$P(x + z \leq t) \geq P(x \leq t - \delta \text{ and } z \leq \delta)$$

$$\geq P(x \leq t - \delta) - P(z \geq \delta) \geq f(t - \delta) - \frac{\epsilon}{\delta^2}.$$

LEMMA 2. *Let $\{x_i\}$, $i = 1, 2, \cdots$ be a sequence of independent random variables with mean 0 bounded kth absolute moment, $k > 2$, and variance σ_i^2. Let $M > 0$ and* $\limsup_{n\to\infty} \frac{\sum \sigma_i^2}{n} \le M^2$. *Form the sequence of random variables* $y_n = \frac{x_1 + \cdots + x_n}{M\sqrt{n}}$ *then for any $\epsilon > 0$ and any $t > \bar{t} > 0$*

$$P(y_n \le -t) \le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-t} e^{-\frac{1}{2}x^2} dx + \epsilon \quad \text{for} \quad n \ge N(\epsilon, \bar{t}). \tag{8}$$

PROOF. Form a sequence m_α with $\lim_{\alpha\to\infty} m_\alpha = 0$. Let $y_n = x_n^\alpha + z_n^\alpha$

where

$$x_n^\alpha = \frac{\sum^\alpha x_i}{M\sqrt{n}}, \qquad z_n^\alpha = \frac{\sum x_i - \sum^\alpha x_i}{M\sqrt{n}}.$$

$\sum^\alpha$ denotes summation over all i for which $\sigma_i^2 \ge m_\alpha$ and all sums extend from one to n.

Let f_n^α be the distribution of x_n^α then by LEMMA 1

$$f_n^\alpha(-t + \delta) + \frac{m_\alpha^2}{M\delta^2} \ge P(y_n \le -t) \ge f_n^\alpha(-t - \delta) - \frac{m_\alpha^2}{M\delta^2}.$$

Now we distinguish two cases.

1st Case. The number of integers i with $\sigma_i^2 \ge m_\alpha$ is for some α of order n. In this case $\{f_n^\alpha\}$ differs arbitrarily little from a sequence of normal distributions with mean 0 and the upper limit of the variances at most 1.

2nd Case. The number of integers i with $\sigma_i^2 \ge m_\alpha$ is for every α of smaller order than n. In this case x_n^α converges stochastically to 0. In both cases (8) holds true since m_α can be chosen arbitrarily small.

We can now prove (5). It follows easily from Tschebycheff's theorem that (5) is true if $\lambda = \frac{1}{2}$. Hence we may assume $\lambda < \frac{1}{2}$. Let z_i be defined as at the beginning of this section. Form

$$v_j^k = \sum_{(j-1)k+1}^{jk-1} \frac{2(z_i - E(z_i))}{\sqrt{(n-1)(1-4\lambda^2)}},$$

$$u_j^k = \frac{2(z_{jk} - E(z_{jk}))}{\sqrt{(n-1)(1-4\lambda^2)}}, \qquad \bar{v}_n^k = \sum_{i=m'+1}^{i=n-1} \frac{2(z_i - E(z_i))}{\sqrt{(n-1)(1-4\lambda^2)}}$$

where $m' = gk$ is the largest integer multiple of k which does not exceed $(n - 1)$. We form further

$$x_n^k = \sum_{j=1}^{j=g} v_j^k, \qquad z_n^k = \sum_{j=1}^{j=g} u_j^k.$$

Since $\sigma_{z_n^k}^2 \le \frac{\frac{1}{4}(g-1)}{\frac{1}{4}(n-1)(1-4\lambda^2)} \le \frac{1}{k(1-4\lambda^2)}$ it follows from LEMMA 1 that the distribution of $\frac{2(S - E(S))}{\sqrt{(n-1)(1-4\lambda^2)}}$ differs arbitrarily little from the distribu-

tion of x_n^k for sufficiently large n and k The second and the third absolute moment of $\sqrt{n-1}\ v_j^k$ are bounded. Hence $\sqrt{n-1}\ v_j^k$ fulfills the conditions of LEMMA 2. The application of LEMMA 2 yields (5) and consequently 6.

The integer $N(\epsilon, H, \bar{t})$ is independent of t provided the lower limit of the integral does not exceed $-\bar{t}$. Hence we have proved

THEOREM. *Let $t_1, t_2, \cdots$ be any sequence of numbers satisfying the condition*

$$-\bar{t}_n = \frac{t_n\sqrt{n+1} - 2(n-1)\lambda_n\sqrt{3}}{\sqrt{(3n-3)(1-4\lambda'^2)}} \leq -\bar{t} < 0,$$

where $\lambda' = \liminf_{n\to\infty} \lambda_n$ *if* $\liminf_{n\to\infty} \lambda_n < \frac{1}{2}$ *and* $0 \leq \lambda' < \frac{1}{2}$ *otherwise. Let* $P_n(S, H)$ *be the power of the* S *test with respect to the alternative* H *and critical region* $S \geq \frac{n-1}{2} + t_n\sqrt{\frac{n+1}{12}}$. *Then*

$$\liminf_{n\to\infty}\left[P_n(S_1H) \Big/ \frac{1}{\sqrt{2\pi}}\int_{t_n}^{\infty} e^{-\frac{1}{2}x^2}\,dx\right] \geq 1. \tag{9}$$

It is worthwhile to remark that (9) is sharp. That is to say there exist alternatives for which the left side of (9) is equal to (1). This is obviously the case for any alternative with $P(x_i > x_{i+1}) = \frac{1}{2} + \lambda$ and $P(z_i = 1 \text{ and } z_{i+1} = 1) = P(z_i = 1)\cdot P(z_{i+1} = 1)$. These conditions are, for instance fulfilled by the alternative given by $P(x_{i+1} = a - \delta - \cdots - \delta^i) = \frac{1}{2} + \lambda$, $P(x_{i+1} = C + \delta + \cdots + \delta^i) = \frac{1}{2} - \lambda, i = 1, 2, \cdots$ where $(a - c) > \dfrac{2\delta}{1-\delta} > 0$.

If $t_n = t$ for every n then (9) implies the consistency of the test if the order of λ_n is larger than $1/\sqrt{n}$. It may also be seen that the test is not consistent with respect to alternatives for which λ_n is of order at most equal to $\dfrac{1}{\sqrt{n}}$. This remark refers of course only to alternatives for which x_i is independent of x_j for $i \neq j$.

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THE ASYMPTOTIC DISTRIBUTION OF RUNS OF CONSECUTIVE ELEMENTS

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In a permutation of $1, 2, \cdots, n$ let r denote the number of instances in which i is next to $i+1$, i.e., in which either of the successions $(i, i+1)$ or $(i+1, i)$ occurs. Thus for the permutation 234651, $r = 3$. In [3] Wolfowitz[1] has proposed the use of r for significance tests in the non-parametric case, and in [4] he has shown that asymptotically r has the Poisson distribution with mean value 2. It is to be noted that $W(R)$, the number of runs as defined by Wolfowitz, is equal to $n - r$.

In this note we shall derive more explicit results concerning the asymptotic distribution of r. In a random permutation (all permutations being regarded as equally probable) let the probability of exactly r successions as above be $P(n, r)$, and let $M(n, k)$ denote the k-th factorial moment of the distribution, that is

$$M(n, k) = \Sigma_r r(r-1) \cdots (r-k+1)P(n, r)$$

We shall show that

$$(1) \qquad M(n, k) = 2^k\left[1 - \frac{k+1}{2k}\binom{k}{1}\frac{k}{n} + \frac{k+2}{2^2 k}\binom{k}{2}\frac{k(k-1)}{n(n-1)} - \cdots\right]$$

$$(2) \qquad P(n, r) = \frac{2^r e^{-2}}{r!}\left[1 - \frac{r^2 - 3r}{2n} + \frac{r^4 - 8r^3 + 9r^2 + 22r - 16}{8n(n-1)}\right] + 0(n^{-3}).$$

Since 2^k is the k-th factorial moment of the Poisson distribution with mean 2, either of these results serves to verify the asymptotic Poisson character of the distribution of r.

It would be possible to obtain some kind of explicit formula for the general term of (2), but there seems to be no reasonably simple form.

Proof of (1). Let A_i denote the event "$i+1$ comes right after i" and B_i the event "i comes right after $i+1$" $(i = 1, \cdots, n-1)$.. The joint probability of k of these $2n - 2$ events is either 0, if they are incompatible, or $(n-k)!/n!$ if they are compatible—for in the latter case we in effect assign positions for k of the elements and are then free to permute the $n - k$ others Let $f(n, k)$ denote the number of ways of selecting k compatible events. Then it is known that ([1], eq. (40))

$$(3) \qquad M(n, k) = k!f(n, k)(n-k)!/n! = f(n, k)\Big/\binom{n}{k}.$$

[1] I am indebted to Dr Wolfowitz for calling my attention to this problem, and to its identity with what I called the "n-kings problem" in [2].

The relations of incompatibility can be summarized by the statement that A_i is incompatible with B_j if $|i - j| \leq 1$. In view of (3), our task thus reduces to the proof of the following combinatorial lemma

LEMMA. *Suppose $2n - 2$ objects $A_1, \cdots, A_{n-1}, B_1, \cdots, B_{n-1}$ are given. Let $f(n, k)$ denote the number of ways of selecting k objects with the restriction that A_i and B_j must not both be chosen when $|i - j| \leq 1$. Then*

$$(4) \qquad \frac{f(n, k)}{2^k} = \sum_{i=0}^{k} (-1)^i \frac{k+1}{2^i k} \binom{k}{i} \binom{n-i}{k-i}.$$

PROOF. We split the acceptable selections into two subsets: those which include A_{n-1} and those which do not. Let the latter be $g(n, k)$ in number. Since the selections which include A_{n-1} must omit B_{n-1} and B_{n-2}, it is clear that they are $g(n - 1, k - 1)$ in number. Thus

$$(5) \qquad f(n, k) = g(n, k) + g(n - 1, k - 1).$$

Similarly we split the selections which omit A_{n-1} according as they omit or include B_{n-1}; we obtain

$$(6) \qquad g(n, k) = f(n - 1, k) + g(n - 1, k - 1).$$

Elimination of g from (5) and (6) yields[2]

$$(7) \qquad f(n, k) = f(n - 1, k) + f(n - 1, k - 1) + f(n - 2, k - 1).$$

We can now make an inductive proof of (4) Assuming (4), we have

$$\frac{f(n, k) - f(n - 1, k)}{2^k} = \Sigma(-1)^i \frac{k+1}{2^i k} \binom{k}{i} \binom{n-i-1}{k-i-1}$$

$$\frac{f(n-2, k-1)}{2^{k-1}} = \Sigma(-1)^i \frac{k+i-1}{2^i(k-1)} \binom{k-1}{i} \cdot \left[\binom{n-i-1}{k-i-1} - \binom{n-i-2}{k-i-2}\right]$$

$$= \Sigma(-1)^i \binom{n-i-1}{k-i-1} \left[\frac{k+i-1}{2^i(k-1)} \binom{k-1}{i} + \frac{k+i-2}{2^{i-1}(k-1)} \binom{k-1}{i-1}\right].$$

In view of the identity

$$\frac{k+i}{k} \binom{k}{i} = \frac{k+i-1}{k-1} \binom{k-1}{i} + \frac{k+i-2}{k-1} \binom{k-1}{i-1}$$

we now readily verify that the right hand side of (4) satisfies (7). To complete the induction we must check the appropriate boundary conditions. According to (4) we have

$$\frac{f(k, k)}{2^k} = \sum_{i=0}^{k} (-1)^i \frac{k+i}{2^i k} \binom{k}{i} = 0,$$

$f(n, 1) = 2n - 2$, both as they should be

[2] This recursion formula is essentially the same as equation (20) in [2].

Note There are various other formulas for $f(n, k)$; we have selected (4) as it exhibits the asymptotic behaviour best. In an unpublished investigation John Riordan obtained a neat representation as a hypergeometric function:

$$f(n, k) = 2(n - k)F(1 - k, 1 + k - n; 2; 2)$$

and derived corresponding recursion formulas. Essentially the same result was given by Wolfowitz [3]. Still another formula given by Riordan is

$$f(n, k) = 2\sum_{i=0}^{k-1} \binom{k-1}{i}\binom{n-1-i}{k}.$$

A symbolic version is given in §5 of [2].

Proof of (2). From the formula of Poincaré ([1], eq. (29))

$$r!P(n, r) = \sum_{k=r}^{n} (-1)^{k+r} M(n, k)/(k - r)!$$

or, in a cabalistic symbolic form, $P(n, r) = M^r e^{-M}/r!$. We substitute the successive terms of (1) and we may let the sum run to infinity at a cost of $O(n^{-m})$ for any positive m. The first term contributes[3]

$$\sum_{k=r}^{\infty} (-1)^{k+r} 2^k/(k - r)! = 2^r \sum_{i=0}^{\infty} (-2)^i/i! = 2^r e^{-2}.$$

Again since

$$k^2 + k = (k - r)(k - r - 1) + (2r + 2)(k - r) + r^2 + r,$$

the next term yields

$$\sum_{k=r}^{\infty} (-1)^{k+r}(k^2 + k)2^{k-1}/(k - r)! = 2^r e^{-2}\left(2 - 2r - 2 + \frac{r^2 + r}{2}\right),$$

and so on in obvious fashion.

Some indication of the asymptotic behavior of $P(n, r)$ is afforded by the following table for $n = 10$. It is to be noted that, because of the form of (2), the approach to Poisson is much more rapid for $r = 0$ and 3 than for other r.

r	$P(10, r)$	Poisson	First two terms of (2)
0	.132	.135	.135
1	.300	.271	.298
2	.305	.271	.298
3	.179	.180	.180
4	.065	.090	.072
5	.015	.036	.018
6	.002	.012	.001
7	.000	.003	.001

[3] My thanks are due to Mr. Riordan for correcting an error in this section, and for many helpful suggestions concerning the entire paper.

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ON THE APPROXIMATE DISTRIBUTION OF RATIOS

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The purpose of this paper is to apply Cramer's theorem of asymptotic expansion[1] and Berry's theorem[2] to study the approximate distribution of ratios of the following two types:

$$\text{(I)} \qquad Z = \frac{1}{n}(Y_1 + \cdots + Y_n) \Big/ \frac{1}{m}(X_1 + \cdots + X_m) = \bar{Y}/\bar{X},$$

$$\text{(II)} \qquad Z = Y \Big/ \frac{1}{m}(X_1 + \cdots + X_m) = Y/\bar{X}.$$

In (I) the X_i, Y_j are independent, the Y_j are equi-distributed,[3] and the X_i are equi-distributed and positive. In (II) $X_1, \cdots, X_n$, Y are independent and positive, and the X_i are equi-distributed

1. The ratio (I). Assume that (I1) the absolute kth moment of X_i and that of Y_j are finite and positive, where k is a fixed integer ≥ 3,
(I2) the distribution of X_i and that of Y_j are non-singular.

Let

$$\xi = \epsilon(X_i), \qquad \eta = \epsilon(Y_j), \qquad \sigma^2 = \epsilon(X_i^2) - \xi^2, \qquad \tau^2 = \epsilon(Y_j^2) - \eta^2$$

and

$$U = \frac{\sqrt{m}}{\sigma}(\bar{X} - \xi), \qquad V = \frac{\sqrt{n}}{\tau}(\bar{Y} - \eta).$$

Let $F(x)$, $G(x)$ and $H(x)$ be respectively the distribution functions of Z, U and V. Let

$$b = \left(\frac{\sigma^2 x^2}{m} + \frac{\tau^2}{n}\right)^{\frac{1}{2}}, \qquad u = \frac{\xi n - \eta}{b}.$$

Then the relation $Z \leq x$ is equivalent to

$$-\frac{x\sigma U}{b\sqrt{m}} + \frac{\tau V}{b\sqrt{n}} \leq u.$$

[1] H. Cramér, *Random Variables and Probability Distributions* (1937), Chap. 7.

[2] A. C. Berry, "The accuracy of the Gaussian approximation to the sum of independent variates", *Trans. Amer. Math. Soc.*, Vol. 49 (1941), pp. 122-136.

[3] The Y_j are said to be equi-distributed if all Y_j have the same distribution function.

For simplicity we shall assume $x > 0$; the results are, however, general. Then the distribution functions of $-\frac{x\sigma U}{b\sqrt{m}}$ and $\frac{\tau V}{b\sqrt{n}}$ are

$$Pr\left\{-\frac{x\sigma U}{b\sqrt{m}} < y\right\} = 1 - G\left(-\frac{b\sqrt{m}y}{\sigma x}\right), \qquad Pr\left\{\frac{\tau V}{b\sqrt{n}} \le y\right\} = H\left(\frac{b\sqrt{n}y}{\tau}\right).$$

Hence, by the theorem of convolution,

$$(1) \qquad F(x) = \int_{-\infty}^{\infty}\left\{1 - G\left(-\frac{b\sqrt{m}(u-y)}{\sigma x}\right)\right\} dH\left(\frac{b\sqrt{n}y}{\tau}\right).$$

Here we recall the theorems of Cramér and Berry: Under the conditions (I1) and (I2)

$$(2) \qquad G(x) = \Phi(x) + \sum_{\nu=1}^{k-3}\frac{P_\nu(x)}{m^{\nu/2}} + \frac{D_k}{m^{\frac{1}{2}(k-2)}},$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,dy, \qquad P_\nu(x) = \sum_{j=1}^{\nu} c_{j\nu}\Phi^{(\nu+2j)}(x),$$

and $| D_k |$ is less than a positive number which depends only on k and the distribution of X_i. If $k = 3$, condition (I2) may be removed[4]

Analogously,

$$(3) \qquad H(x) = \Phi(x) + \sum_{\nu=1}^{k-3}\frac{Q_\nu(x)}{n^{\nu/2}} + \frac{D_k'}{m^{\frac{1}{2}(k-2)}},$$

where

$$Q_\nu(x) = \sum_{j=1}^{\nu} d_{j\nu}\Phi^{(\nu+2j)}(x).$$

In the sequel we shall use the letter Δ_k to denote an unspecified quantity such that $| \Delta_k |$ is less than a positive number which depends only on k, the distribution of X_i and the distribution of Y_j.

Using (2) we have

$$(4) \qquad 1 - G(-x) = \Phi(x) + \sum_{\nu=1}^{k-3}\frac{(-1)^\nu P_\nu(x)}{m^{\nu/2}} + \frac{D_k}{m^{\frac{1}{2}(k-2)}}$$

and this making this substitution in (1) we get

$$F(x) = \int_{-\infty}^{\infty}\Phi\left(\frac{b\sqrt{m}(u-y)}{\sigma x}\right) dH\left(\frac{b\sqrt{n}y}{\tau}\right)$$
$$+ \sum_{\nu=1}^{k-3}\frac{(-1)^\nu}{m^{\nu/2}}\int_{-\infty}^{\infty}P_\nu\left(\frac{b\sqrt{m}(u-y)}{\sigma x}\right) dH\left(\frac{b\sqrt{n}y}{\tau}\right) + \frac{\Delta_k}{m^{\frac{1}{2}(k-2)}},$$

[4] This last assertion constitutes Berry's theorem.

and so by partial integration,

$$F(x) = \int_{-\infty}^{\infty} H\left(\frac{b\sqrt{n}(u-y)}{\tau}\right) d\Phi\left(\frac{b\sqrt{m}y}{\sigma x}\right) + \sum_{\nu=1}^{k-3} \frac{(-1)^\nu}{m^{\nu/2}} \int_{-\infty}^{\infty} H\left(\frac{b\sqrt{n}(u-y)}{\tau}\right) dP_\nu\left(\frac{b\sqrt{m}y}{\sigma x}\right) + \frac{\Delta_k}{m^{\frac{1}{2}(k-2)}}.$$

Making the transformation $y = \sigma x v/b\sqrt{m}$ and writing

$$\alpha = \frac{b\sqrt{n}}{\tau}, \qquad \beta = \frac{\sigma\sqrt{n}\,x}{\tau\sqrt{m}} \tag{5}$$

we get

$$\begin{aligned} F(x) &= \int_{-\infty}^{\infty} H(du - \beta v)\Phi'(v)\,dv + \sum_{\nu=1}^{k-3}\frac{(-1)^\nu}{m^{\nu/2}}\int_{-\infty}^{\infty} H(\alpha u - \beta y)P_\nu'(v)\,dv + \frac{\Delta_k}{m^{\frac{1}{2}(k-2)}} \\ &= I_0 + \sum_{\nu=1}^{k-3}\frac{(-1)^\nu}{m^{\nu/2}} I_\nu + \frac{\Delta_k}{m^{\frac{1}{2}(k-2)}}. \end{aligned}$$

For I_0 we use (3) and obtain

$$I_0 = \int_{-\infty}^{\infty} \Phi(\alpha u - \beta v)\Phi'(v)\,dv + \sum_{\nu=1}^{k-3}\frac{1}{n^{\nu/2}}\int_{-\infty}^{\infty} Q_\nu(\alpha u - \beta v)\Phi'(v)\,dv + \frac{\Delta_k}{n^{\frac{1}{2}(k-2)}}.$$

For I_ν we use (3) with k replaced by $k - \nu$. Thus

$$I_\nu = \int_{-\infty}^{\infty} \Phi(\alpha u - \beta v)P_\nu'(v)\,dv + \sum_{\mu=1}^{k-3-\nu}\frac{1}{n^{\mu/2}}\int_{-\infty}^{\infty} Q_\mu(\alpha u - \beta v)P_\nu'(v)\,dv + \frac{\Delta_k}{n^{\frac{1}{2}(k-2-\nu)}}.$$

Combining these results we get

$$\begin{aligned} F(x) = \int_{-\infty}^{\infty} \Phi(\alpha u - \beta v)\Phi'(v)\,dv &+ \sum_{\nu=1}^{k-3}\frac{1}{n^{\nu/2}}\int_{-\infty}^{\infty} Q_\nu(\alpha u - \beta v)\Phi'(v)\,dv \\ &+ \sum_{\nu=1}^{k-3}\frac{(-1)^\nu}{m^{\nu/2}}\int_{-\infty}^{\infty}\Phi(\alpha u - \beta v)P_\nu'(v)\,dv \\ &+ \sum_{\nu=1}^{k-3}\sum_{\mu=1}^{k-3-\nu}\frac{(-1)^\nu}{m^{\nu/2}n^{\mu/2}}\int_{-\infty}^{\infty} Q_\mu(\alpha u - \beta u)P_\nu'(v)\,dv + R_k, \end{aligned} \tag{6}$$

where

$$R_k = \frac{\Delta_k}{m^{\frac{1}{2}(k-2)}} + \frac{\Delta_k}{n^{\frac{1}{2}(k-2)}} + \sum_{\nu=1}^{k-3}\frac{\Delta_k}{m^{\nu/2}n^{\frac{1}{2}(k-2-\nu)}} = \Delta_k\left(\frac{1}{\sqrt{m}} + \frac{1}{\sqrt{n}}\right)^{k-2}.$$

Now by (5), $\alpha > 0$ and $\alpha^2 - \beta^2 = 1$. For such values of α and β, however, it follows easily from the theorem of convolution that

$$\int_{-\infty}^{\infty}\Phi(\alpha u - \beta v)\Phi'(v)\,dv = \Phi'(u).$$

As differentiation under the integration sign is justified by the boundedness of the derivatives of Φ we have

$$\alpha^p \int_{-\infty}^{\infty} \Phi^{(p)}(\alpha u - \beta v)\Phi'(v)\, dv = \Phi^{(p)}(u).$$

Repeated partial integration then gives

$$\int_{-\infty}^{\infty} \Phi^{(p)}(\alpha u - \beta v)\Phi^{(q)}(v)\, dv = \beta^{q-1} \int_{-\infty}^{\infty} \Phi^{(p+q-1)}(\alpha u - \beta v)\Phi'(v)\, dv$$
$$= \frac{\beta^{q-1}}{\alpha^{p+q-1}} \Phi^{(p+q-1)}(u).$$

Hence

$$\int_{-\infty}^{\infty} Q_\nu(\alpha u - \beta v)\Phi'(v)\, dv = \sum_{j=1}^{\nu} d_{j\nu} \int_{-\infty}^{\infty} \Phi^{(\nu+2j)}(\alpha u - \beta v)\Phi'(v)\, dv$$
$$= \sum_{j=1}^{\nu} \frac{d_{j\nu}}{\alpha^{\nu+2j}} \Phi^{(\nu+2j)}(u),$$

$$\int_{-\infty}^{\infty} \Phi(\alpha u - \beta v)P_\nu'(v)\, dv = \sum_{j=1}^{\nu} c_{j\nu} \int_{-\infty}^{\infty} \Phi(\alpha u - \beta v)\Phi^{(\nu+2j+1)}(v)\, dv$$
$$= \sum_{j=1}^{\nu} \frac{\beta^{\nu+2j} c_{j\nu}}{\alpha^{\nu+2j}} \Phi^{(\nu+2j)}(u),$$

$$\int_{-\infty}^{\infty} Q_\mu(\alpha u - \beta v)P_\nu'(v)\, dv = \sum_{i=1}^{\mu} \sum_{j=1}^{\nu} d_{i\mu} c_{j\nu} \int \Phi^{(\mu+2i)}(\alpha u - \beta v)\Phi^{(\nu+2j+1)}(v)\, dv$$
$$= \sum_{i=1}^{\mu} \sum_{j=1}^{\nu} d_{i\mu} c_{j\nu} \frac{\beta^{\nu+2j}}{\alpha^{\mu+\nu+2i+2j}} \Phi^{(\mu+\nu+2i+2j)}(u).$$

Making all these substitutions in (6) we obtain the final result

$$F(x) = \Phi(u) + \sum_{\nu=1}^{k-3} \frac{(-1)^\nu}{m^{\nu/2}} \sum_{j=1}^{\nu} \frac{\beta^{\nu+2j} c_{j\nu}}{\alpha^{\nu+2j}} \Phi^{(\nu+2j)}(u) + \sum_{\nu=1}^{k-3} \frac{1}{n^{\nu/2}} \sum_{j=1}^{\nu} \frac{d_{j\nu}}{\alpha^{\nu+2j}} \Phi^{(\nu+2j)}(u)$$
$$+ \sum_{\nu=1}^{k-3} \sum_{\mu=1}^{k-3-\nu} \frac{(-1)^\nu}{m^{\nu/2} n^{\mu/2}} \sum_{i=1}^{\mu} \sum_{j=1}^{\nu} d_{i\mu} c_{j\nu} \frac{\beta^{\nu+2j}}{\alpha^{\mu+\nu+2i+2j}} \phi^{(\mu+\nu+2i+2j)}(u)$$
$$+ \Delta_k \left(\frac{1}{\sqrt{m}} + \frac{1}{\sqrt{n}}\right)^{k-2}.$$

If $k = 3$, the result remains true without the condition (I2).

2. The ratio (II). Here we make the following assumptions:
(II1) The kth moment of X_i is finite and positive, where k is a fixed integer $\geq k$, $\epsilon(X_i) = 1$,[5] $\epsilon(X_i^2) - 1 = \sigma^2$.
(II2) The distribution of X_i is non-singular.

[5] As the case $\epsilon(X_i) = 0$ is excluded, there is no loss of generality in this assumption.

Let $U = \sqrt{m}(\bar{X} - 1)/\sigma$, and $F(x)$, $G(x)$ and $H(x)$ be respectively the distribution functions of Z, U and Y. Then

$$F(x) = Pr\left\{Y - \frac{\sigma x U}{\sqrt{m}} \leq x\right\}$$

Because of the positiveness of X, and Y we may always assume $x > 0$. Then, by the theorem of convolution,

$$F(x) = \int_{-\infty}^{\infty}\left\{1 - G\left(-\frac{\sqrt{m}(x-y)}{\sigma x}\right)\right\} dH(y).$$

Using (4) we have

$$F(x) = \int_{-\infty}^{\infty}\left\{\Phi\left(\frac{\sqrt{m}(x-y)}{\sigma x}\right) + \sum_{\nu=1}^{k-3}\frac{(-1)^{\nu}}{m^{\nu/2}} P_{\nu}\left(\frac{\sqrt{m}(x-y)}{\sigma x}\right)\right\} dH(y) + \frac{A_k}{m^{\frac{1}{2}(k/2)}},$$

where, as throughout the rest of this paper, A_k represents an unspecified quantity such that $|A_k|$ is less than a positive number depending only on k, the distribution of X, and the distribution of Y. By partial integration we get

$$(7)\quad F(x) = \int_{-\infty}^{\infty} H(x-y)\, d\left\{\Phi\left(\frac{\sqrt{m}\,y}{\sigma x}\right) + \sum_{\nu=1}^{k-3}\frac{(-1)^{\nu}P_{\nu}\left(\frac{\sqrt{m}\,y}{\sigma x}\right)}{m^{\nu/2}}\right\} + \frac{A_k}{m^{\frac{1}{2}(k-2)}}$$

$$= \int_{-\infty}^{\infty} H\left(x - \frac{\sigma x z}{\sqrt{m}}\right)\left(\Phi'(z) + \sum_{\nu=1}^{k-3}\frac{(-1)^{\nu}P_{\nu}'(z)}{m^{\frac{1}{2}(k-2)}}\right) dz + \frac{A_k}{m^{\frac{1}{2}(k-2)}}.$$

An interesting special case is the following: Suppose that (II3) $H^{(k-2)}(x)$ exists and is continuous for all $x \geq 0$; (II4) the functions

$$\xi_{\nu}(x) = x^{\nu}H^{(\nu)}(x) \qquad (\nu = 1, \cdots, k-3)$$

are bounded, i.e.

$$\xi_{\nu}(x) = A_k;$$

(II3) there is a positive constant $c < 1$ such that

$$x^{k-2}H^{(k-2)}(y) = A_k$$

for all $x \geq 0$ and $(1-c)x \leq y \leq (1+c)x$. Under these conditions we have

$$H\left(x - \frac{\sigma x z}{\sqrt{m}}\right) = \sum_{\nu=0}^{k-3}\frac{(-1)^{\nu}\sigma^{\nu}x^{\nu}z^{\nu}H^{(\nu)}(x)}{\nu!\, m^{\nu/2}}$$
$$+ \frac{(-1)^{k-2}\sigma^{k-2}x^{k-2}z^{k-2}}{(k-2)!\, m^{\frac{1}{2}(k-2)}} H^{(k-2)}\left(x + \frac{\delta\sigma x z}{\sqrt{m}}\right) \quad (|\delta| \leq 1),$$

and so, for $|z| \leq \dfrac{c\sqrt{m}}{\sigma}$ we have

$$(8)\qquad H\left(x - \frac{\sigma x z}{\sqrt{m}}\right) = \sum_{\nu=0}^{k-3}\frac{(-1)^{\nu}\sigma^{\nu}\xi_{\nu}z^{\nu}}{\nu!\, m^{\nu/2}} + \frac{A_k z^{k-2}}{m^{\frac{1}{2}(k-2)}}.$$

Separate now the integral in (7) into two parts:

$$I_1 = \int_{|z| \leq c\sqrt{m}/\sigma}, \qquad I_2 = \int_{|z| > c\sqrt{m}/\sigma}.$$

Now

$$|I_2| \leq \int_{|z| > c\sqrt{m}/\sigma} \left| \Phi'(z) + \sum_{\nu=1}^{k-3} \frac{(-1)^\nu P'_\nu(z)}{m^{\nu/2}} \right| dz.$$

Evidently this last integral is exponially small and so is $A_k/m^{\frac{1}{2}(k-2)}$. By (8),

$$I_1 = \int_{|z| \leq c\sqrt{m}/\sigma} \left(\sum_{\nu=0}^{k-3} \frac{(-1)^\nu \sigma^\nu \xi_\nu z^\nu}{\nu!\, m^{\nu/2}} \right) \left(\Phi'(z) + \sum_{\nu=1}^{k-3} \frac{(-1)^\nu P'_\nu(z)}{m^{\nu/2}} \right) dz + \frac{A_k}{m^{\frac{1}{2}(k-2)}}$$

$$= \int_{-\infty}^{\infty} \left(\sum_{\nu=0}^{k-3} \frac{(-1)^\nu \sigma^\nu \xi_\nu z^\nu}{\nu!\, m^{\nu/2}} \right) \left(\Phi'(z) + \sum_{\nu=1}^{k-3} \frac{(-1)^\nu P'_\nu(z)}{m^{\nu/2}} \right) dz + \frac{A_k}{m^{\frac{1}{2}(k-2)}}.$$

Combining these results we obtain

$$F(x) = \int_{-\infty}^{\infty} \left(\sum_{\nu=0}^{k-3} \frac{(-1)^\nu \sigma^\nu \xi_\nu z^\nu}{\nu!\, m^{\nu/2}} \right) \left(\Phi'(z) + \sum_{\nu=1}^{k-3} \frac{(-1)^\nu}{m^{\nu/2}} \sum_{j=1}^{\nu} c_{j\nu} \Phi^{(\nu+2j+1)}(z) \right) dz + \frac{A_k}{m^{\frac{1}{2}(k-2)}}$$

$$= \sum_{\nu=0}^{k-3} \frac{d_\nu \xi_\nu}{m^{\nu/2}} I_{\nu 1} + \sum_{\nu=0}^{k-3} \sum_{\mu=1}^{k-3} \sum_{j=1}^{\mu} \frac{d_{j\mu\nu} \xi_\nu}{m^{\frac{1}{2}(\mu+\nu)}} \Phi_{\nu,\, \mu+2j+1} + \frac{A_k}{m^{\frac{1}{2}(k-2)}}$$

$$= \sum_1 + \sum_2 + \frac{A_k}{m^{\frac{1}{2}(k-2)}},$$

where

$$I_{\alpha\beta} = \int_{-\infty}^{\infty} z^\alpha \Phi^{(\beta)}(z)\, dz.$$

Now the following facts can easily be established by means of partial integration:

(9) $$I_{\alpha\beta} = 0 \quad \text{when} \quad \alpha - \beta \text{ is even},$$

(10) $$I_{\alpha\beta} = 0 \quad \text{when} \quad \beta - \alpha > 1.$$

By (9), the non-vanishing terms in $\sum_1$ are the even terms and the non-vanishing terms in $\sum_2$ are those for which $\mu + \nu$ is even. Hence

$$\sum_1 = \sum_{\nu=0}^{[\frac{1}{2}(k-3)]} \frac{e_\nu \xi_{2\nu}}{m^\nu},$$

$$\sum_2 = \sum_{\nu=0}^{[\frac{1}{2}(k-3)]} \sum_{\mu=1}^{[\frac{1}{2}(k-3)]} \sum_{j=1}^{2\mu} \frac{e_{j\mu\nu} \xi_{2\nu}}{m^{\mu+\nu}} I_{2\nu,\, 2\mu+2j+1} + \sum_{\nu=0}^{[\frac{1}{2}(k-4)]} \sum_{\mu=0}^{[\frac{1}{2}(k-4)]} \sum_{j=1}^{2\mu+1} \frac{e'_{j\mu\nu} \xi_{2\nu+1}}{m^{\mu+\nu+1}} I_{2\nu+1,\, 2\mu+2j+2}.$$

Using (10) to reduce $\sum_2$ further we get

$$\begin{aligned}
\sum_2 &= \sum_{\nu=2}^{[\frac{1}{2}(k-3)]} \sum_{\mu=1}^{\nu-1} \sum_{j=1}^{2\mu} \frac{e_{j\mu\nu}\,\xi_{2\nu}}{m^{\mu+\nu}} I_{2\nu,\,2\mu+2j+1} + \sum_{\nu=1}^{[\frac{1}{2}(k-4)]} \sum_{\mu=0}^{\nu-1} \sum_{j=1}^{2\mu+1} \frac{e'_{j\mu\nu}\,\xi_{2\nu+1}}{m^{\mu+\nu+1}} I_{2\nu+1,\,2\mu+2j+2} \\
&= \sum_{\nu=0}^{[\frac{1}{2}(k-7)]} \sum_{\mu=0}^{\nu} \frac{g_{\mu\nu}\,\xi_{2\nu+4}}{m^{\mu+\nu+3}} + \sum_{\nu=0}^{[\frac{1}{2}(k-6)]} \sum_{\mu=0}^{\nu} \frac{g'_{\mu\nu}\,\xi_{2\nu+3}}{m^{\mu+\nu+2}} \\
&= \sum_{\alpha=0}^{[\frac{1}{2}(k-9)]} \frac{1}{m^{\alpha+3}} \sum_{\beta=[\frac{1}{2}(\alpha+1)]}^{\alpha} h_{\alpha\beta}\,\xi_{2\beta+4} + \sum_{\alpha=0}^{[\frac{1}{2}(k-6)]} \frac{1}{m^{\alpha+2}} \sum_{\beta=[\frac{1}{2}(\alpha+1)]}^{\alpha} h'_{\alpha\beta}\,\xi_{2\beta+3} + \frac{A_k}{m^{\frac{1}{2}(k-2)}} \\
&= \sum_{i=3}^{[\frac{1}{2}(k-3)]} \frac{1}{m^i} \sum_{j=[\frac{1}{2}(i-2)]}^{i-3} l_{ij}\,\xi_{2j+4} + \sum_{i=2}^{[\frac{1}{2}(k-3)]} \frac{1}{m^i} \sum_{j=[\frac{1}{2}(i-1)]}^{i-2} l'_{ij}\,\xi_{2j+3} + \frac{A_k}{m^{\frac{1}{2}(k-2)}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_1 + \sum_2 = \xi_0 + \frac{e_1\xi_2}{m} + \frac{e_2\xi_4 + l'_{20}\xi_3}{m^2} + \sum_{\nu=3}^{[\frac{1}{2}(k-3)]} \frac{1}{m^\nu} \\
\cdot\left(e_\nu\xi_{2\nu} + \sum_{\mu=[\frac{1}{2}(\nu-2)]}^{\nu-3} l_{\mu\nu}\,\xi_{2\nu+4} + \sum_{\mu=[\frac{1}{2}(\nu-2)]}^{\nu-2} l'_{\mu\nu}\,\xi_{2\nu+3}\right) + \frac{A_k}{m^{\frac{1}{2}(k-2)}} \\
= \xi_0 + \sum_{\nu=1}^{k-3} \frac{1}{m^\nu} \sum_{j=\nu+1}^{2\nu} p_{j\nu}\,\xi_j + \frac{A_k}{m^{\frac{1}{2}(k-2)}}.
\end{aligned}$$

Hence

$$F(x) = \xi_0 + \sum_{\nu=1}^{k-3} \frac{1}{m^\nu} \sum_{j=\nu+1}^{2\nu} p_{j\nu}\,\xi_j + \frac{p_k}{m^{\frac{1}{2}(k-2)}}.$$

Our final conclusion is: Under the conditions (II1)–(II5) formula (11) is true; if $k = 3$, (11) remains true without the condition (II2).

ON THE DISTRIBUTION OF THE SERIAL CORRELATION COEFFICIENT

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The distribution of the serial correlation coefficient, in samples drawn from a parent distribution with zero serial correlation, has been studied by many authors. Anderson [1] obtained the exact distribution. Dixon [3] and Koopmans [4] have given approximate distributions, each attained by smoothing the characteristic values of the numerator of $\bar{r}$ in (1) below. Dixon smoothed the characteristic values in the generating function and obtained his results by comparing the moments of the exact distribution with those of the approximation, of which the first T are found to be exact. Koopmans smoothed the characteristic values in the exact distribution function. Here we evaluate Koopmans result and show that it is the same as Dixon's approximation. It thus appears that in this case it is immaterial whether the characteristic values are smoothed before or after inverting the characteristic function. We also add Tables comparing confidence limits for the exact distribution, for the approximation referred to, and for a normal approximation.

We define the serial correlation coefficient as

$$\bar{r} = \frac{\sum_{t=1}^{T} x_t x_{t+1}}{\sum_{t=1}^{T} x_t^2}, \quad x_{T+1} = x_1. \tag{1}$$

Then Koopmans obtains, if the true value ρ of $\bar{r}$ equals 0, and the x_t are normally and independently distributed with mean 0 and variance σ^2, the approximate distribution $T/2 - 2$.

$$\bar{h}(\bar{r}, T) = \frac{2^{\frac{1}{2}T}(\frac{1}{2}T - 1)}{\pi} \int_0^{\arccos \bar{r}} (\cos \alpha - \bar{r})^{\frac{1}{2}T-2} \sin \tfrac{1}{2}T\alpha \sin \alpha \, d\alpha. \tag{2}$$

Although in the distribution problem T is a positive integer, it is useful to consider the right-hand member of (2) as the definition of $\bar{h}(\bar{r} \cdot T)$ for those complex values of T for which it exists.

Let $R(T)$ denote the real part of T. If $R(T) > 2N + 2$, we obtain

$$\begin{aligned} \frac{d^N}{d\bar{r}^N} \bar{h}(\bar{r}, T) = {} & \frac{(-1)^N 2^{\frac{1}{2}T}(\frac{1}{2}T - 1)}{\pi} (\tfrac{1}{2}T - 2)(\tfrac{1}{2}T - 3) \cdots (\tfrac{1}{2}T - N - 1) \\ & \cdot \int_0^{\arccos \bar{r}} (\cos \alpha - \bar{r})^{\frac{1}{2}T-2-N} \sin \tfrac{1}{2}T\alpha \sin \alpha \, d\alpha. \end{aligned} \tag{3}$$

Now, according to [2], tables 41, 42.

$$\begin{aligned} & \int_0^{\pi/2} (\cos \alpha)^{\frac{1}{2}T-2-N} \sin \tfrac{1}{2}T\alpha \sin \alpha \, d\alpha \\ & \qquad = \frac{\frac{1}{2}T\pi}{2^{\frac{1}{2}T-N}} \frac{\Gamma(\frac{1}{2}T - N - 1)}{\Gamma(\frac{1}{2}(T - N + 1))\Gamma(\frac{1}{2}(1 - N))}. \end{aligned} \tag{4}$$

Deonote by $\bar{h}^{(N)}(0, T)$ the value of $\frac{d^N}{d\bar{r}^N}\,\bar{h}(\bar{r}, T)$ for $\bar{r} = 0$. Then for $R(T) > 2N + 2$,

$$\bar{h}^{(N)}(0, T) = \frac{(-1)^N 2^N \Gamma(\frac{1}{2}T + 1)}{\Gamma(\frac{1}{2}(T - N + 1))\Gamma(\frac{1}{2}(1 - N))}. \tag{5}$$

$\bar{h}(\bar{r}, T)$ is analytic in $\bar{r}$ for $|\bar{r}| < 1$, $R(T) > 2$, and is analytic in T for $|\bar{r}| < 1$, $R(T) > 2$. It follows by Hartogs's theorem [5] that $\bar{h}(\bar{r}, T)$ is analytic in $\bar{r}$ and T for $|\bar{r}| < 1$, $R(T) > 2$. By analytic continuation we get that (5) holds for $R(T) > 2$. Consequently

(6) If N is odd, $\bar{h}^{(N)}(0, T) = 0$;

(7) if N is even,

$$\frac{\bar{h}^{(r)}(0, T)}{\bar{h}(0, T)} = \frac{2^N \Gamma(\frac{1}{2}T + \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}(T - N + 1))\Gamma(\frac{1}{2}(1 - N))}.$$

Let $N = 2P$, then

$$\begin{aligned}
&\frac{1}{(2P)!}\,\frac{\bar{h}^{(2P)}(0, T)}{\bar{h}(0, T)} \\
&= \frac{(-1)^P 2^{2P}}{(2P)!}\cdot\left(\frac{T-1}{2}\right)\cdot\left(\frac{T-3}{2}\right)\cdots\left(\frac{T-2P+1}{2}\right)\left(\frac{1\cdot 3\cdots(2P-1)}{2^P}\right) \\
&= \frac{(-1)^P}{(2P)!}\left(\frac{T-1}{2}\right)\left(\frac{T-3}{2}\right)\cdots\left(\frac{T-2P+1}{2}\right)\frac{(2P)!}{P!} \\
&= \frac{1}{P!}\left[\frac{d^P}{\{d(\bar{r}^2)\}^P}\,(1 - \bar{r}^2)^{\frac{1}{2}(T-1)}\right]_{\bar{r}=0}.
\end{aligned} \tag{8}$$

According to (5)

$$\bar{h}(0, T) = \frac{\Gamma(\frac{1}{2}T + 1)}{\Gamma(\frac{1}{2}T + \frac{1}{2})\Gamma(\frac{1}{2})} = \frac{1}{\int_{-1}^{1} (1 - \bar{r}^2)^{\frac{1}{2}(T-1)}\, d\bar{r}}. \tag{9}$$

Hence

$$\bar{h}(\bar{r}, T) = \frac{\Gamma(\frac{1}{2}T + 1)(1 - \bar{r}^2)^{\frac{1}{2}(T-1)}}{\Gamma(\frac{1}{2}T + \frac{1}{2})\Gamma(\frac{1}{2})}, \tag{10}$$

which is the same as Dixon's expression (3.22).

A more elementary proof by complete induction for integral values of T can be based on the recurrent differential equation (14) which is of interest in itself. To this end we shall write (2) in a different form which is easily obtained through partial integration.

$$\bar{h}(\bar{r}, T) = \frac{2^{\frac{1}{2}T}\cdot\frac{1}{2}T}{\pi}\int_0^{\operatorname{arc}\cos\bar{r}} (\cos\alpha - \bar{r})^{\frac{1}{2}T-1}\cos\tfrac{1}{2}T\,\alpha\, d\alpha. \tag{11}$$

Differentiating with respect to $\bar{r}$,

$$\bar{h}'(\bar{r},\ T) = -\frac{\frac{1}{2}T(\frac{1}{2}T - 1)2^{\frac{1}{2}T}}{\pi} \int_0^{\operatorname{arc\,cos} \bar{r}} (\cos \alpha - \bar{r})^{\frac{1}{2}T-2} \cos \tfrac{1}{2}\, T\alpha \, d\alpha.$$

$$= -\frac{\frac{1}{2}T(\frac{1}{2}T - 1)2^{\frac{1}{2}T}}{\pi} \int_0^{\operatorname{arc\,cos} \bar{r}} (\cos \alpha - \bar{r})^{\frac{1}{2}T-2} (\cos \tfrac{1}{2}(T - 2)\, \alpha \cos \alpha$$

$$- \sin \tfrac{1}{2}(T - 2)\, \alpha \sin \alpha)\, d\alpha$$

$$= -\frac{\frac{1}{2}T(\frac{1}{2}T - 1)2^{\frac{1}{2}T}}{\pi} \int_0^{\operatorname{arc\,cos} \bar{r}} (\cos \alpha - \bar{r})^{\frac{1}{2}T-1} \cos \tfrac{1}{2}(T - 2)\alpha \, d\alpha \tag{12}$$

$$- \frac{\frac{1}{2}T(\frac{1}{2}T - 1)2^{\frac{1}{2}T}}{\pi} \int_0^{\operatorname{arc\,cos} \bar{r}} (\cos \alpha - \bar{r})^{\frac{1}{2}T-2} \cos \tfrac{1}{2}(T - 2)\alpha \, d\alpha$$

$$+ \frac{\frac{1}{2}T(\frac{1}{2}T - 1)2^{\frac{1}{2}T}}{\pi} \int_0^{\operatorname{arc\,cos} \bar{r}} (\cos \alpha - \bar{r})^{\frac{1}{2}T-2}$$

$$\cdot \sin \tfrac{1}{2}(T - 2)\alpha \sin \alpha \, d\alpha$$

$$= -\frac{\frac{1}{2}T(\frac{1}{2}T - 1)2^{\frac{1}{2}T}\bar{r}}{\pi} \int_0^{\operatorname{arc\,cos} \bar{r}} (\cos \alpha - \bar{r})^{\frac{1}{2}(T-2)-1} \tag{13}$$

$$\cdot \cos \tfrac{1}{2}(T - 2)\alpha \, d\alpha,$$

because the first and third terms in (12) cancel as may be shown by integrating by parts.

Hence (13) reduces to the recurrent differential equation

$$\bar{h}'(\bar{r},\ T) = -2 \cdot \tfrac{1}{2}T\bar{r}\bar{h}(\bar{r},\ T - 2). \tag{14}$$

Let us now assume that

$$\bar{h}(\bar{r},\ T - 2) = \frac{\Gamma(\frac{1}{2}T)}{\Gamma(\frac{1}{2}T - \frac{1}{2})\Gamma(\frac{1}{2})} (1 - \bar{r}^2)^{\frac{1}{2}(T-3)}. \tag{15}$$

Then (14) becomes

$$\bar{h}(\bar{r},\ T) = -2\bar{r} \cdot \tfrac{1}{2}T \frac{\frac{1}{2}(T - 1)}{\frac{1}{2}T - \frac{1}{2}} \frac{\Gamma(\frac{1}{2}T)}{\Gamma(\frac{1}{2}T - \frac{1}{2})\Gamma(\frac{1}{2})} (1 - \bar{r}^2)^{\frac{1}{2}T-3} \tag{16}$$

$$= -2\bar{r} \cdot \tfrac{1}{2}(T - 1) \frac{\Gamma(\frac{1}{2}T + 1)}{\Gamma(\frac{1}{2}T + \frac{1}{2})\Gamma(\frac{1}{2})} (1 - \bar{r}^2)^{\frac{1}{2}(T-1)-1}.$$

Integrating, one obtains

$$\bar{h}(\bar{r},\ T) = \frac{\Gamma(\frac{1}{2}T + 1)}{\Gamma(\frac{1}{2}T + \frac{1}{2})\Gamma(\frac{1}{2})} (1 - \bar{r}^2)^{\frac{1}{2}(-2)}. \tag{17}$$

No constant of integration occurs because (17) agrees with (5) for $\bar{r} = 0$ and $N = 0$.

It remains to prove the validity of (17) for the initial values $T = 3$ and $T = 4$. If $T = 4$

$$
\begin{aligned}
\bar{h}(\bar{r}, 4) &= \frac{4}{\pi} \int_0^{\arccos \bar{r}} \sin 2\alpha \sin \alpha \, d\alpha \\
&= \frac{8}{3\pi} \sin^3 \alpha \Big|_0^{\arccos \bar{r}} = \frac{8}{3\pi} (1 - \bar{r}^2)^{\frac{3}{2}} = \frac{\Gamma(3)}{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})} (1 - \bar{r}^2)^{\frac{1}{2}(4-1)}.
\end{aligned} \tag{18}
$$

For $T = 3$,

$$
\bar{h}(\bar{r}, 3) = \frac{\frac{1}{2}2^{\frac{1}{2}}}{\pi} \int_0^{\arccos \bar{r}} \frac{\sin \frac{3}{2}\alpha \sin \alpha}{\sqrt{\cos \alpha - \bar{r}}} \, d\alpha. \tag{19}
$$

Substitute $\cos \alpha = \bar{r} + (1 - \bar{r}) \sin^2 \theta$. We get

$$
\begin{aligned}
\bar{h}(\bar{r}, 3) &= \frac{2(1 - \bar{r})}{\pi} \int_0^{\frac{1}{2}\pi} \{(1 + 2\bar{r}) \cos^2 \theta + 2(1 - \bar{r}) \sin^2 \theta \cos^2 \theta\} \, d\theta \\
&= \tfrac{3}{4}(1 - \bar{r}^2) = \frac{\Gamma(\frac{5}{2})}{\Gamma(2)\Gamma(\frac{1}{2})} (1 - \bar{r}^2)^{\frac{1}{2}(3-1)}
\end{aligned} \tag{20}
$$

which completes the proof.

A short table of confidence limits is included, corresponding to the 5% and 1% significance levels, comparing the exact distribution given by Anderson [1] (the values in parentheses being graphically interpolated by him), the distribution (10), and the normal curve with the same mean and standard deviation.

Confidence limits for $\bar{r}$

T	5%			1%		
	Exact	(10)	Normal	Exact	(10)	Normal
3	.854	.729	.736	.970	.882	1.040
4	.713	.669	.672	.898	.833	.950
5	.622	.621	.622	.823	.789	.879
6	.570	.582	.582	.762	.750	.823
7	.545	.549	.548	.714	.715	.775
8	(.521)	.521	.520	(.682)	.685	.736
9	.498	.497	.496	.656	.658	.701
10	(.477)	.476	.475	(.633)	.634	.672
11	.457	.458	.456	.612	.612	.645
15	.400	.400	.399	.543	.543	.564
20	(.351)	.352	.351	(.480)	.482	.496
25	.317	.317	.317	.437	:37	.448
30	(.291)	.291	.291	(.404)	.403	.411
35	(.271)	.271	.270	(.377)	.376	.382
40	(.255)	.254	.254	(.355)	.354	.359
45	.240	.240	.240	.335	.335	.339

It is thus seen that the distribution (10) provides satisfactory significance levels for $T \geq 9$ whereas the normal approximation provides satisfactory 5% significance levels for the same range. The normal approximation appears to be unsatisfactory, however, at the 1% significance level even for T as high as 45. The normal approximation here used is not the same as that used by Anderson ([1], p. 53), which assumes $\frac{\sqrt{T}\,\bar{r}}{\sqrt{1+2\bar{r}^2}}$ to be normally distributed.

The following table shows a comparison between a few more confidence limits of the Type II curve (10) and the normal curve with same first two moments for a few values of T.

Confidence limits for $\bar{r}$

T	5%		4%		3%		2%		1%	
	(10)	Normal	(10)	Normal	(10)	Normal	(10)	Normal	(10)	Normal
15	.400	.399	.423	.425	.452	.456	.488	.498	.543	.564
20	.352	.351	.373	.373	.398	.401	.431	.438	.482	.496
25	.317	.317	.336	.337	.360	.362	.390	.395	.437	.448

REFERENCES

[1] R. L. Anderson, *Serial Correlation in the Analysis of Time Series*, unpublished thesis, Iowa State College, 1941

[2] D Bierens de Haan, *Nouvelles Tables d'Integrales Definies*, Leyden, 1867.

[3] T. Koopmans, "Serial Correlation and Quadratic Forms in Normal Variables", *Annals of Math Stat.*, Vol 13 (1942), pp. 14–33

[4] W. J. Dixon, "Further contributions to the problem of serial correlation", *Annals of Math Stat.*, Vol. 14 (1944).

[5] W F. Osgood, *Lehrbuch der Functionentheorie*, Vol. 2. Part 2, Leipzig, 1907.

NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

A NOTE CONCERNING HOTELLING'S METHOD OF INVERTING A PARTITIONED MATRIX

BY F. V. WAUGH

War Food Administration, Washington

Professor Hotelling recently presented several methods of computing the inverse of a matrix.[1] Among these was a method of partitioning a square matrix of $2p$ rows into four square matrices, a, b, c and d, of p rows each, resulting in the partitioned matrix,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The inverse of this matrix can also be written as a partitioned matrix,

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix}.$$

Then, multiplying the original matrix by its inverse we get four matrix equations,

$$\begin{aligned} aA + bB &= 1 \qquad & aC + bD &= 0 \\ cA + dB &= 0 \qquad & cC + dD &= 1. \end{aligned}$$

These equations can be solved for A, B, C, and D.

Professor Hotelling's solution requires the inversion of four p-rowed matrices. It is possible, however, to solve these equations by formulas involving only two inversions. The formulas are

$$\begin{aligned} D &= (d - ca^{-1}b)^{-1} \qquad & B &= -Dca^{-1} \\ C &= -a^{-1}bD \qquad & A &= a^{-1} - a^{-1}bB. \end{aligned}$$

As an example of the procedure let the given matrix be

$$\left[\begin{array}{rr|rr} 26 & -10 & 15 & 32 \\ 19 & 45 & -14 & -8 \\ \hline -12 & 16 & 27 & 13 \\ 32 & 29 & -35 & 28 \end{array}\right].$$

[1] HAROLD HOTELLING "Some new methods of matrix calculation," *Annals of Math. Stat*, Vol 14 (1943), pp 1-34.

The necessary steps in computation are

$$a^{-1} = \begin{bmatrix} .03309 & .00735 \\ -.01397 & .01912 \end{bmatrix} \qquad a^{-1}b = \begin{bmatrix} .39345 & 1.00008 \\ -.47723 & -.60000 \end{bmatrix}$$

$$ca^{-1} = \begin{bmatrix} -.62060 & .21772 \\ .65375 & .78968 \end{bmatrix} \qquad ca^{-1}b = \begin{bmatrix} -12.35708 & -21.60096 \\ -1.24927 & 14.60256 \end{bmatrix}.$$

Note that a convenient check at this point is to compute both $(ca^{-1})b$ and $c(a^{-1}b)$

$$d - ca^{-1}b = \begin{bmatrix} 39.35708 & 34.60096 \\ -33.75073 & 13.39744 \end{bmatrix}$$

$$(d - ca^{-1}b)^{-1} = D = \begin{bmatrix} .00790 & -.02041 \\ .01991 & .02322 \end{bmatrix}$$

$$-a^{-1}bD = C = \begin{bmatrix} -.02302 & -.01519 \\ .01572 & .00419 \end{bmatrix}$$

$$-Dca^{-1} = B = \begin{bmatrix} .01825 & .01440 \\ -.00282 & -.02267 \end{bmatrix}$$

$$a^{-1} - a^{-1}bB = A = \begin{bmatrix} .02873 & .02436 \\ -.00696 & .01239 \end{bmatrix}.$$

The last four of these matrices are the four parts of the inverse, which can be written

$$\begin{bmatrix} .02873 & .02436 & -.02302 & -.01519 \\ -.00696 & .01239 & .01572 & .00419 \\ .01825 & .01440 & .00790 & -.02041 \\ -.00282 & -.02267 & .01991 & .02322 \end{bmatrix}.$$

The accuracy of the computations can be checked by multiplying the original matrix by the computed inverse matrix. The product should, of course, be a close approximation of the identity matrix. If further accuracy is called for we can use Hotelling's iterative formula,

$$C_1 = C_0(2 - AC_0)$$

where C_0 is the estimated inverse; A is the original matrix; and C_1 is a second approximation of the inverse.

NEWS AND NOTICES

Readers are invited to submit to the Secretary of the Institute news items of interest

Personal Items

Professor W. G. Cochran of Iowa State College has gone overseas as a consultant for the United States War Department.

Professor A. R. Crathorne of the University of Illinois has retired with the title of Professor Emeritus.

Professor William Feller of Brown University has been appointed Professor of Mathematics at Cornell University, Ithaca, New York, as of July 1, 1945.

Associate Professor Joe J. Livers has returned to Montana State College at Bozeman after receiving his doctorate in February at the University of Michigan.

Assistant Professor W. A. Vezeau of the University of Detroit has been appointed Assistant Professor of Mathematics at St. Louis University.

Associate Professor S. S. Wilks of Princeton University has been promoted to a professorship.

The American Statistical Association elected ten Fellows during 1944. Of these ten, five are members of the Institute. They are A. E. Brandt, W. G. Cochran, Gertrude M. Cox, Alan Treloar, and Sewall Wright. The President of the Association is Dr. Walter A. Shewhart, a charter member of the Institute and its President during 1944.

New Members

The following persons have been elected to membership in the Institute:

Allendoerfer, Asso. Prof. Carl B. Ph.D. (Princeton) Haverford College, Haverford, Pa.

Beckstead, Lt. (j.g.) Gordon L. M.S. (Michigan) Aerologist, U. S. Navy. Aerology, Navy #151, c/o Fleet Post Office, San Francisco, Calif

Berman, Abraham J. M.A. (Brooklyn) Statistician. *1460 College Avenue, New York, N. Y.*

Bigelow, Julian H. Asso. Director, Statistical Research Group, Columbia University. 401 West 118th St , New York 27, N. Y.

Bowen, Earl K. A.M (Boston) Instr Math. Northeastern Univ , Boston, Mass. On military leave—Scientific Consultant, Office of Field Service, O.S.R D. *6 Sibley Ave., W. Springfield, Mass.*

Canter, Stanley D. B.S. (Coll. City of N Y.) Statistician, Lerner Shops, Inc., New York, N. Y. *2676 Morris Ave., The Bronx, 58, New York, N. Y.*

Cohen, Karl. Ph D. (Columbia) Physicist, Standard Oil Development Co. Esso Laboratories, Research Division, P. O. Box 243, Elizabeth B, N. J.

Cooper, William W. A B. (Chicago) Instr. in Economics, University of Chicago. 6539 S. Ellis Ave., Chicago 37, Ill.

Davidson, James H. B.S. (Norwich Univ.) Research Physicist, Hercules Powder Co. *Box 344, Christiansburg, Va.*

Epstein, Benjamin Ph.D. (Illinois) Staff Assistant, Westinghouse Electric & Mfg. Co., Quality Control Dept., Rm. 3-A-17, East Pittsburgh, Pa.

Gauthier, Prof. Abel A.M. (Columbia) Prof. of Mathematics, Université de Montreal, 2900 Mount Royal Blvd., Montreal, Canada.

Gersten, Lydia Blumenthal B.A. (Hunter) Res. Stat. *1001 Lincoln Place, Brooklyn 13, N. Y.*

Goffman, Casper Ph.D. (Ohio State) Staff Asst., Quality Control Dept., Westinghouse Elec. & Mfg. Co., Rm. 3-A-17, East Pittsburgh, Pa.

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James, R. W. M.A. (Toronto) Asst. to Director, Washington Div., Wartime Prices & Trade Board. Room 3068, Railroad Retirement Bldg., Washington, D. C.

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Ladd, Robert Boyd M.A. (Texas Coll. of Arts & Industries) Stat. Consultant, OCT, Transport Economics, Traffic Control Div., War Dept., Washington, D. C. *903 Wade Ave., Rockville, Md.*

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Lewis, Wyatt H. B.S. (Calif. Inst. of Tech.) Quality Control Engineer. *212 East H Street, Ontario, Calif.*

Mathisen, Ensign Harold C. A B. (Princeton) Ensign, USNR. *59 Fernwood Road, East Orange, N. J.*

Miller, Robert Carmi Res. Engineer, Elgin National Watch Co., Elgin, Ill.

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Stigler, Prof. George J. Ph.D. (Chicago) Prof. of Economics, Member, Res. Staff, National Bureau of Econ. Res., University of Minnesota, Minneapolis, Minn.

Weingarten, Harry M.A. (Columbia) Math. Teacher, School of Aviation Trades *1330 Morris Ave., Bronx 56, N. Y*

Weinstein, Joseph M.S. (C.C N. Y.) Res. Analyst, Vacuum Tube Tests & Standardization, Camp Evans Signal Lab. Signal Corps. *13 Washington Village, Asbury Park, N. J.*

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Young, Captain Chen-Pang B.A. (National Tsing Hua Univ., China) Ordnance Dept., Chinese Army. *2311 Massachusetts Ave., N. W., Washington 8, D. C.*

Corrections to the Directory Published in the December 1944 Issue

The name of Dr. Walter Schilling was omitted from the Directory. It should have appeared as follows:

Schilling, Walter M.D. (Harvard) Asst. Clinical Professor of Medicine. Stanford University Hospital, San Francisco 15, California

The name of Professor Godfrey H. Thomson, Director of the Training of Teachers, University of Edinburgh, Edinburgh, Scotland, was misspelled.

CHOICE OF ONE AMONG SEVERAL STATISTICAL HYPOTHESES

By Ralph J. Brookner[1]

New York City

1. Introduction. Statistical decision is a term which we will apply to that phase of statistical inference which deals with the following question. Consider one or several variates whose distribution function depends on one or several unknown parameters; suppose there be given a finite number of mutually exclusive hypotheses regarding the parameters, whose totality completely exhausts every possibility. If a sample of observations on the variates is made, the choice of one of the given hypotheses on the basis of that sample is called a statistical decision. In other words, to make a statistical decision is to give a procedure which will divide the sample space into as many regions as there are given hypotheses, and to set up a one-to-one correspondence between these regions and the hypotheses so that if the sample point lies in any particular region, the corresponding hypothesis is chosen

This notion is quite closely connected with both of the fields of statistical inference that have engaged most of the modern statistical theorists On the one hand, it may be considered a generalization of the notion of testing hypotheses, for in this theory, one gives a procedure which divides the sample space into a region of rejection and a region of non-rejection of a given null hypothesis. Then one makes either of two decisions depending upon which of the regions contains the sample point On the other hand, the theory of estimation is a generalization of the notion of statistical decision in which the number of alternatives is not restricted to be finite

As in any phase of statistical inference, our primary aim is to define broad principles upon which "good" or "best" procedures for making statistical decisions may be based. The general problem of statistical decisions has been formulated by A. Wald, who has also proposed a principle on which the solution can be based. We are interested, however, in several of the simpler but important particular problems in which quite serious calculation difficulties are encountered in actually finding Wald's solution Hence, we will propose in its stead another principle which quite closely resembles Wald's for selecting a solution of the problem of statistical decision.

It may be pointed out immediately that, from a purely logical point of view, the substitute principle we shall offer will probably be considered to be less acceptable than its predecessor. We will find, however, by considering its application to some of the well known problems of testing hypotheses, that the principle is at least reasonable in leading to certain well accepted results.

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[1] Research under a grant-in-aid of the Carnegie Corporation of New York

2. Principle determining the "best" procedure. We will first discuss briefly Wald's principle and the definition of the criterion that we will employ will be accomplished by pointing out the differences. A much more general formulation is possible [1], [2], but we will discuss the principle as it will be directly applied to the problems of statistical decisions when the number of hypotheses is finite

Consider the variates $x_1, x_2, \cdots, x_p$ whose probability density function $f(x_1, x_2, \cdots, x_p \mid \theta_1, \theta_2, \cdots, \theta_k)$ is known except for the unknown values of the parameters $\theta_1, \theta_2, \cdots, \theta_k$. We denote by θ a point in k-dimensional space whose coordinates are $(\theta_1, \theta_2, \cdots, \theta_k)$ and shall speak of this parameter space as Ω. Suppose that ω is any subset of Ω and that S represents a system of finitely many such sets which are mutually disjunct and which cover Ω. Each element, ω_0, of S corresponds to a hypothesis H_{ω_0}, which is the hypothesis that θ is a point of ω_0, and the system of all such hypotheses corresponding to S we denote by H_S.

A sample of N observations on $x_1, x_2, \cdots, x_p$ is drawn and the sample may be considered as a point, E, in the pN dimensional sample space; denote the sample space by M. We want to decide on the basis of the point E which of the hypotheses of H_S should be accepted. That is, we seek a procedure by which the sample space may be divided into a system of mutually exclusive regions M_ω which are the same in number as the number of elements of S, and by which a correspondence is set up so that the falling of the sample point into a particular M_{ω_0} shall cause us to accept a particular hypothesis H_{ω_0} as the true one. If the totality of regions M_ω be denoted M_S, it is necessary to give a principle by which we may prefer a particular system M_S over any other system M'_S.

Wald introduces the notion of a weight function of errors, a function of the parameters and of the decision made, which might well be defined as the loss incurred if θ be the true parameter point and the sample point falls in M_ω which causes us to accept the hypothesis H_ω. Denote the weight function by $W(\theta, \omega_E)$ where ω_E stands for that hypothesis which we choose if E is the sample point; then we require that $W(\theta, \omega_E)$ be non-negative, and if θ lies in ω_E, $W(\theta, \omega_E) = 0$ for then the correct decision has been made and there is no loss.

Perhaps the notion of a weight function can be most clearly understood, and its importance appreciated, if we consider the place of statistics in the business world, where possible losses are often computable in terms of money. The weight function may be taken to be equal to this loss. Suppose a manufacturing plant has a process which manufactures a product whose efficiency is a measurable quantity that we will denote by x. Suppose x is a random variable whose distribution depends only upon its mean value θ, and the company contemplates renewing its machinery if the mean value of the efficiency falls short significantly from a particular value θ_0. Then on the basis of a sample of N observations on x, one of two decisions must be reached: the rejection of the hypothesis $\theta \geqq \theta_0$ (the decision to renew the machinery), or the non-rejection of $\theta \geqq \theta_0$ (the decision not to renew it). Suppose the region M_ω is the region of the sample space such

that if E falls into M_ω, we reject $\theta \geqq \theta_0$ and $M_{\bar\omega}$ is the complementary region. Then we may say that the weight function can be defined by

$$\begin{aligned} W(\theta, \bar\omega) &= 0 && \text{for } \theta \geqq \theta_0 \\ W(\theta, \bar\omega) &= g(\theta) && \text{for } \theta < \theta_0 \\ W(\theta, \omega) &= 0 && \text{for } \theta < \theta_0 \\ W(\theta, \omega) &= h(\theta) && \text{for } \theta \geqq \theta_0 \end{aligned}$$

where $h(\theta)$ is the company's monetary loss in needlessly changing its machinery and $g(\theta)$ is a function which expresses the company's loss in not changing its process even though the true value of the parameter is $\theta < \theta_0$. The function $g(\theta)$ may be of almost any form, but it is only reasonable that it should be a monotonic non-decreasing function of $|\theta_0 - \theta|$, since the loss should, it seems, increase as the true value of θ is farther from θ_0.

Wald then defines the risk as the expected value of the loss; since θ is an unknown, the risk will be a function of θ, and it will also be a function of the system M_S:

$$r(\theta, M_S) = \int_M W(\theta, \omega_E)\cdot f(E \mid \theta)\, dE.$$

According to Wald, the "best" system of regions, M_S, is that system for which the maximum of the risk function with respect to the parameter θ is a minimum with respect to all possible systems, M'_S, of regions. Several important properties are enjoyed by the system of regions defined in this way, though other reasonable definitions are possible Perhaps the criterion of minimizing an average with respect to θ of $r(\theta, M_S)$ rather than the maximum may be considered more plausible, but such definitions would raise the question of which average should be used, and the result obtained by using any particular average would not be invariant with respect to transformations of the parameter space.

Using the notations as introduced above, and introducing the notation $W(\theta, \omega_i)$ to be the weight function if the ith hypothesis is chosen, the principle which we will use to solve some of the problems of statistical decisions can be given as follows: In place of the risk function, we consider the s functions

$$R_i(\theta, E) = W(\theta, \omega_i)\cdot f(E \mid \theta) \qquad (i = 1, 2, \cdots, s)$$

where $f(E \mid \theta)$ is a notation for the probability density, and s is the number of given hypotheses. If we denote by $\bar R_i(E)$ the least upper bound of $R_i(\theta, E)$ with respect to θ, then we choose the system of "best" regions of acceptance by *including each sample point E in a region M_i determined such that for all E_0 in M_i, $\bar R_i(E_0) \leqq \bar R_j(E_0)$ for all $j \neq i$.*

It is interesting to note that a rather general case exists in which the principle is exactly equivalent with the test of a hypothesis based upon the likelihood ratio principle. Consider the distribution function $f(x_1, x_2, \cdots, x_p \mid \theta_1, \theta_2, \cdots, \theta_k)$ which is a bounded function of the x's and θ's. Suppose we are interested in the test of the hypothesis $(\theta_1, \theta_2, \cdots, \theta_k) \epsilon\, \omega$ where ω is a closed

set of points of the parameter space which does not contain any open subset of the parameter space Furthermore assume that for each set of x's the distribution function is continuous in $\theta_1, \cdots, \theta_k$ on an open subset of Ω containing ω

We will show that the principle will lead to the test based on the likelihood ratio if the following is the weight function:

I. If ω is accepted, the loss is zero if the true parameter point is in ω, and the loss is a constant c_1 if the true parameter point is not in ω.

II If ω is rejected (i.e. $\bar{\omega}$ is chosen), the loss is zero if the true parameter point is in $\bar{\omega}$ and is a constant c_2 if the true parameter point is in ω.

Consider then the region of the sample space for which ω is rejected according to the principle. This region is that for which

$$\text{l.u.b. w.r.t. } \theta \text{ in } \omega \text{ of } [c_2 f(x \mid \theta)] < \text{l.u.b. w.r.t. } \theta \text{ in } \bar{\omega} \text{ of } [c_1 f(x \mid \theta)]$$

where we have set $f(x \mid \theta) = f(x_1, x_2, \cdots, x_n \mid \theta_1, \theta_2, \cdots, \theta_k)$, and where l.u.b. w.r.t. means "least upper bound with respect to." But the left-hand member of this inequality is equal to

$$c_2[\text{l.u.b. w.r.t. } \theta \text{ in } \omega \text{ of } f(x \mid \theta)]$$

and because of the restriction on ω and the continuity of f, we can see that the l.u.b. of $f(x \mid \theta)$ with respect to all θ in $\bar{\omega}$ must coincide with the l.u.b. of the function with respect to all θ in Ω, which is the total parameter space. Thus we have that the hypothesis ω is rejected when

$$c_2[\text{l.u.b. w.r.t. } \theta \text{ in } \omega \text{ of } f(x \mid \theta)] < c_1[\text{l.u.b. w.r.t. } \theta \text{ in } \Omega \text{ of } f(x \mid \theta)]$$

or when

$$\frac{\text{l.u.b. w.r.t. } \theta \text{ in } \omega \text{ of } f(x \mid \theta)}{\text{l.u.b. w.r.t. } \theta \text{ in } \Omega \text{ of } f(x \mid \theta)} < \frac{c_1}{c_2}.$$

The left hand member of this inequality is the likelihood ratio statistic introduced by Neyman and Pearson [3]; hence our test is exactly equivalent with the likelihood ratio test where the size of the critical region is determined by c_1 and c_2.

We pose the following quite hypothetical example to show circumstances under which the principle proposed is reasonable. The principle does not exactly apply as it was stated in terms of probability densities and the example involves discrete probabilities, but the logic seems somewhat applicable. Suppose a game is played which consists of the player's guessing the number of white balls in an urn known to contain 10 balls, each of which is either white or black, on the basis of a sample of four drawings with replacements from the urn. Let us assume that there are eleven mutually exclusive hypotheses (as to the number of white balls in the urn) to choose among, and the player must make a choice of one of them after observing the drawing which can give 16 different results. Assume that the one who plays the game pays a banker a varying sum of money if he makes a wrong decision and that the banker has the privilege of choosing

the population (i e the number of white and black balls originally in the urn). Now on the basis of the assumption that the banker knows the player's decision function and will attempt to fix the population so as to make the player's expected loss a maximum, it is clear that Wald's principle, which minimizes the maximum loss, leads to the best way to play the game.

Now suppose that instead of one player making the choice among the decisions, we have 16 players participating in the game and the first player is to make the choice if, and only if, the drawing is $WWWW$, the second player if the drawing is $WWWB$, and so on, where W stands for the drawing of a white ball and B for the drawing of a black one. In this case, if player x assumes that the banker will try to choose the population most unfavorable to him, then his decision function based on the new principle is the best method of play.

Although the example indicates that in the usual case which would come up in practice, Wald's principle would lead to the better procedure, since the statistician is usually faced with the necessity of giving a decision no matter what the sample point is, the new principle is useful since one may hope that in many practical cases the two principles will not lead to widely varying results, especially if the sample is large.

3. Application of the criterion to the case of testing the mean of a normal distribution. Now we will show that the criterion will lead to the widely used test of "Student's hypothesis." Suppose x is known to be distributed normally with unknown mean μ and unknown variance σ^2 On the basis of a sample of N independent observations $x_1, x_2, \quad, x_N$, "Student's t" is used to test the hypothesis $\mu = 0$. If $\bar{x}$ is the arithmetic mean of the N observations and s^2 the usual sample estimate of the variance, then with $t = \sqrt{N}\,\bar{x}/s$, the hypothesis is to be rejected if $|t| \geqq t_0$ where t_0 is a critical value at some chosen level of significance α obtained from the distribution of t under the null hypothesis. We will use the notation ω_1 for the set of points $\mu \neq 0$ and ω_2 for the set of points $\mu = 0$.

We will consider the problem in reference to the particular weight function defined as follows:

$$\begin{aligned} W(\mu, \sigma, \omega_2) &= (\mu/\sigma)^k \qquad \text{for } \mu \neq 0 \\ W(0, \sigma; \omega_1) &= W \\ W(\mu, \sigma; \omega_1) &= 0 \qquad\qquad \text{for } \mu \neq 0 \\ W(0, \sigma, \omega_2) &= 0 \end{aligned}$$

where as a matter of convenience, we will take k an even positive integer in order to avoid the introduction of the absolute value of μ/σ which is necessary if k is an odd integer. We also take $k \leqq N$.

The density function of the sample of N observations is

$$\frac{C}{\sigma^N} \cdot e^{-(1/2\sigma^2)S(x_\alpha - \mu)^2}$$

where C is a constant. Then the two functions $R_i(\theta, E)$ are

$$R_1(\theta, E) = \frac{WC}{\sigma^N} \cdot e^{-(1/2\sigma^2)Sx_\alpha^2} \qquad \text{if } \mu = 0$$

$$R_1(\theta, E) = 0 \qquad \text{if } \mu \neq 0$$

$$R_2(\theta, E) = \frac{C\mu^k}{\sigma^{N+k}} \cdot e^{-(1/2\sigma^2)S(x_\alpha-\mu)^2} \qquad \text{if } \mu \neq 0$$

$$R_2(\theta, E) = 0 \qquad \text{if } \mu = 0.$$

To maximize $R_1(\theta, E)$, we set

$$\frac{\partial R_1(\theta, E)}{\partial\sigma} = \left[\frac{-NW}{\sigma^{N+1}} + \frac{WSx_\alpha^2}{\sigma^{N+3}}\right] Ce^{-(1/2\sigma^2)Sx_\alpha^2} = 0$$

which gives

$$\sigma^2 = \frac{Sx_\alpha^2}{N}$$

hence

$$R_1(E) = \frac{CWN^{\frac{1}{2}N}}{(Sx_\alpha^2)^{\frac{1}{2}N}} \cdot e^{-\frac{1}{2}N}.$$

To maximize $R_2(\theta, E)$, we set

$$\frac{\partial R_2(\theta, E)}{\partial\mu} = \left[k + \frac{\mu}{\sigma^2} S(x_\alpha - \mu)\right] \frac{C\mu^{k-1}}{\sigma^{N+k}} \cdot e^{-(1/2\sigma^2)S(x_\alpha-\mu)^2} = 0$$

and

$$\frac{\partial R_2(\theta, E)}{\partial\sigma} = \left[-N - k + \frac{S(x_\alpha - \mu)^2}{\sigma^2}\right] \frac{C\mu^k}{\sigma^{N+k+1}} e^{-(1/2\sigma^2)S(x_\alpha-\mu)^2} = 0$$

which give the two relations

$$\sigma^2 = -\frac{\mu}{k} S(x_\alpha - \mu)$$

and

$$\sigma^2 = \frac{1}{N+k} S(x_\alpha - \mu)^2.$$

Then

$$-\mu(N+k)S(x_\alpha - \mu) = kS(x_\alpha - \mu)^2$$

or

$$\mu^2 - \mu\bar{x}(1 - k/N) - (k/N^2)Sx_\alpha^2 = 0$$

which gives the maximizing value of

$$\mu^* = \frac{\bar{x}(1 - k/N) \pm \sqrt{\bar{x}^2(1 - k/N)^2 + (4k/N^2)Sx_\alpha^2}}{2}$$

and it can easily be shown that the maximum is reached for the value of μ^* using the $+$ sign when $\bar{x}$ is positive and the $-$ sign when $\bar{x}$ is negative. We will carry through the case $\bar{x} > 0$ only as the case $\bar{x} < 0$ follows in a similar manner We have

$$\bar{R}_2(E) = \frac{(\mu^*)^k k^{\frac{1}{2}(N+k)} C}{(\mu^*)^{\frac{1}{2}(N+k)}[-S(x_\alpha - \mu^*)]^{\frac{1}{2}(N+k)}} \cdot e^{-\frac{1}{2}(N+k)}.$$

To find the region of the sample space for which we should accept the hypothesis $\mu \neq 0$ (i.e. the critical region for rejection of the hypothesis $\mu = 0$), we seek those points E for which $\bar{R}_1(E) \leqq \bar{R}_2(E)$, i e. those for which

$$\frac{WN^{\frac{1}{2}N}}{(Sx_\alpha^2)^{\frac{1}{2}N}} \leqq \frac{(\mu^*)^k k^{\frac{1}{2}(N+k)}}{(\mu^*)^{\frac{1}{2}(N+k)}[-S(x_\alpha - \mu^*)]^{\frac{1}{2}(N+k)}} \cdot e^{-\frac{1}{2}k}$$

or for which

$$\frac{(\mu^*)^{\frac{1}{2}(N-k)}[-S(x_\alpha - \mu^*)]^{\frac{1}{2}(N+k)}}{(Sx_\alpha^2)^{\frac{1}{2}N}} \leqq c$$

where c is a positive constant. Since both sides of the inequality are positive, this inequality is equivalent to

$$\text{(1)} \qquad \frac{(\mu^*)^{N-k}(\mu^* - \bar{x})^{N+k}}{(Sx_\alpha^2)^N} \leqq c_1$$

where c_1 is another positive constant.

Now we consider the statistic

$$T^2 = \frac{t^2}{N-1} = \frac{N\bar{x}^2}{Sx_\alpha^2 - N\bar{x}^2} = \frac{N}{Sx_\alpha^2/\bar{x}^2 - N}$$

from which we have

$$Sx_\alpha^2/\bar{x}^2 = (N/T^2) + N.$$

Also note that

$$2(\mu^*/\bar{x}) = (1 - k/N) + \sqrt{(1 - k/N)^2 + (4k/N^2)(Sx_\alpha^2/\bar{x}^2)}$$

(and this is true whether $\bar{x}$ is positive or negative). Now we can write the critical region (1) as

$$\frac{(\mu^*/\bar{x})^{N-k}(\mu^*/\bar{x} - 1)^{N+k}}{(Sx_\alpha^2/\bar{x}^2)^N} \leqq c_1$$

or

$$[1 - k/N + \sqrt{(1 - k/N)^2 + (4k/N)(1 + 1/T^2)}]^{N-k}[1 + 1/T^2]^{-N}$$
$$\cdot[-1 - k/N + \sqrt{(1 - k/N)^2 + (4k/N)(1 + 1/T^2)}]^{N+k} \leqq c_2$$

where c_2 is another positive constant. We denote the left side of this inequality by $\Phi(T^2)$, and it can be shown that $\Phi(T^2)$ is a monotone decreasing function of T^2.

Thus since the critical region is defined by the relation $\Phi(T^2) \leqq$ constant and

the critical region using "Student's t" is $T^2 \geqq$ constant, these procedures are exactly equivalent

4. A Problem in statistical decisions. The question which aroused the interest of the writer in statistical decisions is the following one of multivariate statistical analysis. Suppose $x_1, x_2, \cdots, x_p$ are known to be normally distributed with unknown means and unknown variances and covariances, and on the basis of a set of N independent observations, a test is to be made of the hypothesis $E(x_1) = E(x_2) = \cdots = E(x_p) = 0$ Such a test may be carried out by using the generalized Student Ratio [4], and the hypothesis is either to be rejected or accepted as a whole. But consider the case in which the null hypothesis is rejected; it seems quite natural to ask for a more enlightening statement. Is it not possible to say that on the basis of the sample, the hypothesis should be rejected for $x_{i_1}, x_{i_2}, \cdots, x_{i_k}$ but not rejected for $x_{i_{k+1}}, x_{i_{k+2}}, \cdots, x_{i_p}$? Thus we seek a division of the sample space into 2^p mutually exclusive regions, each of which will lead us to reject the hypothesis of zero expected values for a particular set of the x_i's and to accept it for the remaining set.

We will consider a solution of the problem in the case that the covariance matrix of the joint normal distribution is known, and will motivate that solution by considering first the case of two variables.

Suppose that X and Y are normally and independently distributed with unknown means, α and β, and with unit variances. The joint probability density function is then of the form

$$f(X, Y) = (1/2\pi)\cdot e^{-\frac{1}{2}[(X-\alpha)^2+(Y-\beta)^2]}.$$

The set of hypotheses is given as follows:

H_1 is the hypothesis that $\alpha = 0$ and $\beta = 0$
H_2 is the hypothesis that $\alpha \neq 0$ and $\beta = 0$
H_3 is the hypothesis that $\alpha = 0$ and $\beta \neq 0$
H_4 is the hypothesis that $\alpha \neq 0$ and $\beta \neq 0$.

We have a sample of N independent pairs of observations (X_σ, Y_σ) where $\sigma = 1, 2, \cdots, N$, then the density function in the $2N$ dimensional sample space is

$$(1/2\pi)^N \cdot e^{-\frac{1}{2}S[(X_\sigma-\alpha)^2+(Y_\sigma-\beta)^2]}.$$

We seek the set of regions M_1, M_2, M_3, M_4 in the sample space which are chosen such that if the sample point E falls in M_i, we accept the hypothesis H_i. We take the following as the values of the losses if the wrong decision is reached.

I. If H_1 is accepted,

1) for any parameter point (α, β), the loss is a continuous function of $(\alpha^2 + \beta^2)$, say $W(\alpha^2 + \beta^2)$, which is zero for $\alpha = \beta = 0$, is differentiable, strictly monotonically increasing, and possesses a finite maximum when multiplied by the normal density function.

II. If H_2 is accepted,

i) for any parameter point (α, β) except $(0, 0)$, the loss is $W(\beta^2)$ where W is the same function as above,

ii) the loss is W_1 if the true parameter point is $(0, 0)$.

III. If H_3 is accepted,

i) for any parameter point (α, β) except $(0, 0)$, the loss is $W(\alpha^2)$ where W is the same function as above,

ii) the loss is W_1 if the true parameter point is $(0, 0)$.

IV. If H_4 is accepted,

i) the loss is W_2 if the true parameter point is either $(\alpha, 0)$ for $\alpha \neq 0$, or $(0, \beta)$ for $\beta \neq 0$

ii) the loss is W_3 if the true parameter point is $(0, 0)$

where W_1, W_2, and W_3 are constants subject to some slight restrictions which will be pointed out later.

The functions $R_i(\theta, E)$ are then the following:

$$
\begin{aligned}
R_1(\theta, E) &= W(\alpha^2 + \beta^2)G(\alpha, \beta) && \text{for } \alpha^2 + \beta^2 \neq 0 \\
&= 0 && \text{for } \alpha = \beta = 0 \\
R_2(\theta, E) &= W(\beta^2)G(\alpha, \beta) && \text{for } \beta \neq 0 \\
&= W_1G(0, 0) && \text{for } \alpha = \beta = 0 \\
&= 0 && \text{for } \alpha \neq 0, \beta = 0 \\
R_3(\theta, E) &= W(\alpha^2)G(\alpha, \beta) && \text{for } \alpha \neq 0 \\
&= W_1G(0, 0) && \text{for } \alpha = \beta = 0 \\
&= 0 && \text{for } \alpha = 0, \beta \neq 0 \\
R_4(\theta, E) &= W_2G(\alpha, 0) && \text{for } \alpha \neq 0, \beta = 0 \\
&= W_2G(0, \beta) && \text{for } \alpha = 0, \beta \neq 0 \\
&= W_3G(0, 0) && \text{for } \alpha = \beta = 0 \\
&= 0 && \text{for } \alpha\beta \neq 0
\end{aligned}
$$

where $G(\alpha, \beta)$ is the normal distribution function

$$C \cdot e^{-\frac{1}{2}N[(x-\alpha)^2 + (y-\beta)^2]}$$

x and y being the sample means. It should be pointed out that the use of the distribution of the sample means instead of the joint distribution of the observations is justified since the sample means are sufficient statistics for the parameters α and β.

We will use the notation $\bar{R}_2(E)$ to denote the maximum of $R_2(\theta, E)$ with respect to α and β, and it can easily be seen to be the maximum of two expressions which we will denote by II(1) and II(2) where II(1) is the maximum of $W(\beta^2)G(\alpha, \beta)$ and II(2) is the maximum of $W_1G(0, 0)$. Similarly, $\bar{R}_3(E)$ is the maximum of III(1) and III(2), and $\bar{R}_4(E)$ is the maximum of IV(1), IV(2), and IV(3), where these are the maxima of the two expressions involved in $R_3(\theta, E)$, and the three expressions in $R_4(\theta, E)$, respectively.

We will first show that the function $\bar{R}_1(E)$ is a monotonic increasing function of $(x^2 + y^2)$ We know that the maximum of $R_1(\theta, E)$ is reached for values of

α and β for which the partial derivatives of $R_1(\theta, E)$ with respect to α and β are zero, i.e., for which

$$[N(x-\alpha)W(\alpha^2+\beta^2)+2\alpha W'(\alpha^2+\beta^2)]G(\alpha,\beta)=0$$

and

$$[N(y-\beta)W(\alpha^2+\beta^2)+2\beta W'(\alpha^2+\beta^2)]G(\alpha,\beta)=0$$

where $W'(\alpha^2+\beta^2)$ is the derivative of $W(\alpha^2+\beta^2)$ with respect to $(\alpha^2+\beta^2)$. Since $G(\alpha, \beta) \neq 0$, and $W'(\alpha^2+\beta^2) \neq 0$, these relations imply

$$\begin{vmatrix} x-\alpha & \alpha \\ y-\beta & \beta \end{vmatrix} = 0$$

or $\beta x = \alpha y$. Thus the maximum of the function $R_1(\theta, E)$ occurs for values of α and β which satisfy the relation $\alpha = (x/y)\beta$.

Consider any two straight lines $\alpha = (x'/y')\beta$ and $\alpha = (x''/y'')\beta$, and the values of the function $R_1(\theta, E)$ along these two lines. Obviously the values of the first factor $W(\alpha^2+\beta^2)$ are equal for points along the lines equidistant from the origin. Also, if the values of x', y', x'', and y'' are such that $x'^2+y'^2 = x''^2+y''^2$, the values of the function $G(\alpha, \beta)$ along both lines are equal for points equidistant from the origin, and it follows that $\bar{R}_1(x', y') = \bar{R}_1(x'', y'')$. Thus we have that $\bar{R}_1(E)$ is a function of (x^2+y^2).

Note that if the value of $x''^2+y''^2$ is greater than the value of $x'^2+y'^2$, the curve representing the function $G(\alpha, \beta)$ along $\alpha = (x''/y'')\beta$ is the same as that along the line $\alpha = (x'/y')\beta$, but it is shifted further from the origin. The values of $W(\alpha^2+\beta^2)$ are independent of x and y and the function is monotonic in $\alpha^2+\beta^2$. Thus, the value of $G(\alpha, \beta)$ for which $R_1(\theta, E)$ is a maximum on $\alpha = (x'/y')\beta$ multiplies a larger value of $W(\alpha^2+\beta^2)$ than on $\alpha = (x''/y'')\beta$, so the maximum when $x''^2+y''^2$ exceeds x'^2 is the greater. But this proves that $\bar{R}_1(E)$ is monotonically increasing in (x^2+y^2).

In a similar manner, we now proceed to show that II(1) is a monotonically increasing function of y^2. We know that a necessary condition for a maximum of II(1) is that

$$\frac{\partial \text{II}(1)}{\partial \alpha} = \frac{\partial \text{II}(1)}{\partial \beta} = 0.$$

The first of these two relations is

$$W(\beta^2)N(x-\alpha)G(\alpha,\beta)=0$$

which has the solutions $W(\beta^2) = 0$ and $\alpha = x$. But $W(\beta^2) = 0$ only for $\beta = 0$ and this value is a minimum of II(1), hence we have that the maximum is reached for $\alpha = x$, so

$$\text{II}(1) = \max_{\beta}.\text{ of } W(\beta^2)Ce^{-\frac{1}{2}N(y-\beta)^2}.$$

But along any two lines $\alpha =$ constant in the (α, β)-plane, the function $W(\beta^2)$ has identical monotonically increasing values in β^2 and the normal density

function is identical along two such lines for a fixed value of y^2. An increase in the value of y^2 displaces the normal function from the origin but does not affect its shape, hence the value of the normal density function at which II(1) takes on its maximum is multiplied by a greater value of $W(\beta^2)$ when y^2 is increased, so II(1) is monotonically increasing in y^2. In exactly the same manner, we find that III(1) is a monotonically increasing function of x^2.

Because the remaining functions are identical with the functions considered in the special case above, we have that

$$\begin{aligned} \text{II}(2) &= W_1 C e^{-\frac{1}{2}N(x^2+y^2)} \\ \text{III}(2) &= W_1 C e^{-\frac{1}{2}N(x^2+y^2)} \\ \text{IV}(1) &= W_2 C e^{-\frac{1}{2}Ny^2} \\ \text{IV}(2) &= W_2 C e^{-\frac{1}{2}Nx^2} \\ \text{IV}(3) &= W_3 C e^{-\frac{1}{2}N(x^2+y^2)}. \end{aligned}$$

Now it is apparent that $\bar{R}_1(E)$ is never less than II(1) since

$$W(\alpha^2 + \beta^2)G(\alpha, \beta) \geqq W(\beta^2)G(\alpha, \beta)$$

(the equality holds only for $\alpha = 0$) and since a function which is never less than a second function cannot have a maximum less than the maximum of the second function. Also $\bar{R}_1(E)$ for the same reason is never less than III(1). Thus $\bar{R}_1(E)$ can be the minimum of the four functions $\bar{R}_i(E)$ at most when $\bar{R}_2(E)$ is defined by II(2) and $\bar{R}_3(E)$ is defined by III(2).

Since II(2) and III(2) are the same monotonic decreasing function of $(x^2 + y^2)$ and since $\bar{R}_1(E)$ is a monotonic increasing function of $(x^2 + y^2)$, there is a value r_0^2 of $(x^2 + y^2)$ such that $\bar{R}_1(E) <$ II(2) when and only when $x^2 + y^2 < r_0^2$. But for all values (x, y) we have that $\bar{R}_1(E) \geqq$ II(1) and $\bar{R}_1(E) \geqq$ III(1), hence for all values within the circle $x^2 + y^2 = r_0^2$ we have that

$$\text{II}(1) \leqq \bar{R}_1(E) < \text{II}(2) \tag{2}$$

and

$$\text{III}(1) \leqq \bar{R}_1(E) < \text{III}(2) \tag{3}$$

so it follows that $\bar{R}_2(E)$ is defined by II(2) and $\bar{R}_3(E)$ is defined by III(2) within the circle.

We restrict the values of W_1, W_2, and W_3 used in the definitions of the weight functions to be $W_1 \leqq W_2 \leqq W_3$, hence for all values of (x, y)

$$W_1 C e^{-\frac{1}{2}N(x^2+y^2)} \leqq W_2 C e^{-\frac{1}{2}Ny^2}$$

$$W_1 C e^{-\frac{1}{2}N(x^2+y^2)} \leqq W_2 C e^{-\frac{1}{2}Nx^2}$$

and

$$W_1 C e^{-\frac{1}{2}N(x^2+y^2)} \leqq W_3 C e^{-\frac{1}{2}N(x^2+y^2)}$$

so $\bar{R}_4(E)$ is at least as great as II(2) over the whole plane; hence, in light of relation (2), $\bar{R}_4(E)$ is at least as great as $\bar{R}_1(E)$ for $x^2 + y^2 \leqq r_0^2$. Therefore, since (2) shows that $\bar{R}_1(E) < \bar{R}_2(E)$ within the circle; (3) shows that $\bar{R}_1(E) <$

$\bar{R}_3(E)$ within the circle; and since quite obviously the relations do not hold outside the circle, we have that M_1 is the set of points

$$x^2 + y^2 < r_0^2 .$$

To determine the region M_2, we must determine those points outside M_1 for which $\bar{R}_2(E) < \bar{R}_3(E)$ and $\bar{R}_2(E) < \bar{R}_4(E)$. Consider first the part of the plane outside M_1 for which $\bar{R}_2(E)$ is defined by II(2). This is the region for which II(2) > II(1). Consider the curve in the plane defined by II(2) = II(1), that is,

$$W_1Ce^{-\frac{1}{2}N(x^2+y^2)} = \mathrm{II}(1).$$

We take differentials and have

$$-N(x^2 + y^2)W_1Ce^{-\frac{1}{2}N(x^2+y^2)}[xdx + ydy] = 2y[d\mathrm{II}(1)/d(y^2)]dy$$

but this shows that dy/dx has the opposite sign from y/x since $d\mathrm{II}(1)/d(y^2)$ is always positive. Also note that for $x = 0$, the equation $\bar{R}_1(E) = \mathrm{II}(2)$ is identical with the equation II(1) = II(2), so for $x = 0$, we have II(1) > II(2) when $|y| > r_0$ and II(1) < II(2) when $|y| < r_0$. Furthermore, the curve II(1) = II(2) crosses the x axis at a finite value of x, since for $y = 0$, II(1) is a constant while II(2) is a decreasing function of x.

We will refer to the various regions in the first quadrant of the (x, y)-plane shown in Figure I as follows: A is the part of the quadrant which is M_1; A, B, B', and C are the regions in which $\bar{R}_2(E)$ is defined by II(2), that is, in which II(2) > II(1); and in the same manner, A, B, B', and C' are the regions in which $\bar{R}_3(E)$ is defined by III(2).

Since II(2) and III(2) are identical, we see that within the regions B and B', $\bar{R}_2(E) = \bar{R}_3(E)$ since in these regions $\bar{R}_2(E)$ is defined by II(2) and $\bar{R}_3(E)$ is defined by III(2). We have previously pointed out that II(2) is never greater than $\bar{R}_4(E)$, hence it is clear that B and B' should belong to either M_2 or M_3, and we will arbitrarily decide that B is part of M_2 and B' part of M_3.

Consider then the region C; here $\bar{R}_2(E)$ is defined by II(2) and $\bar{R}_3(E)$ by III(1), so within C

$$\mathrm{II}(2) = \mathrm{III}(2) < \mathrm{III}(1) = \bar{R}_3(E)$$

and again $\mathrm{II}(2) \leqq \bar{R}_4(E)$, so the region C is part of M_2. By the same argument we have that C' is a part of M_3 since within C'

$$\mathrm{III}(2) = \mathrm{II}(2) < \mathrm{II}(1) = \bar{R}_2(E)$$

and $\mathrm{III}(2) \leqq \bar{R}_4(E)$.

Now consider the remainder of the quadrant outside A, B, B', C, and C'. Here $\bar{R}_2(E)$ is defined by II(1) and $\bar{R}_3(E)$ is defined by III(1). Since II(1) is the same monotone increasing function of y^2 as III(1) is of x^2, we have II(1) > III(1) for $|y| > |x|$ and II(1) < III(1) for $|x| > |y|$. Thus we see that in the region under discussion, $\bar{R}_2(E)$ is a minimum at most in the regions D and E and $\bar{R}_3(E)$ a minimum at most in D' and E'.

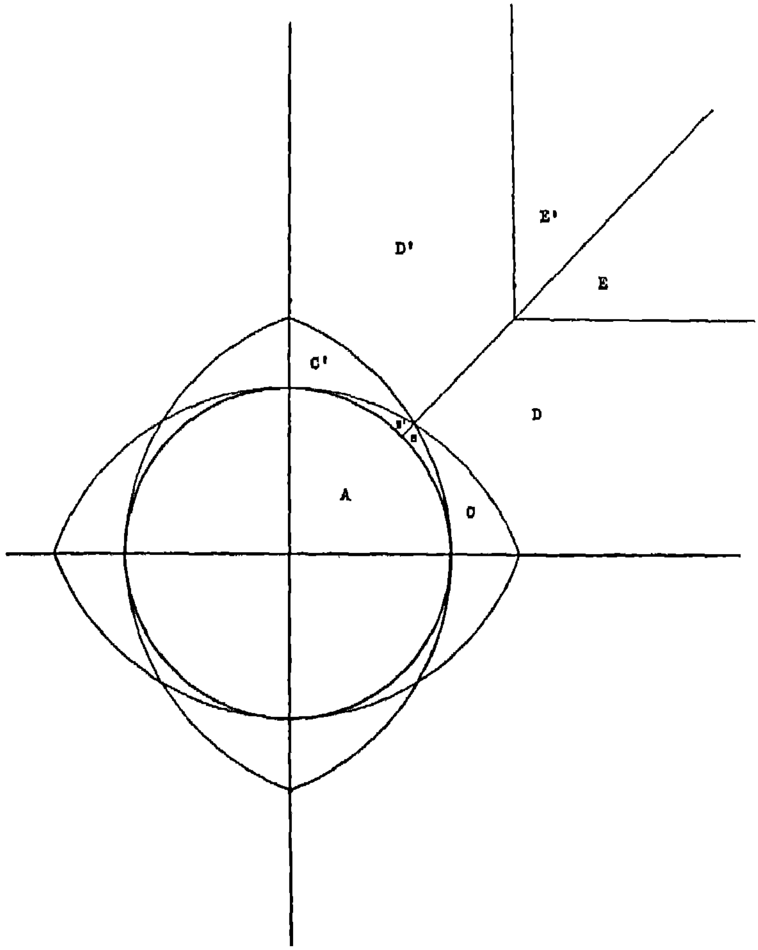

FIG. 1

In order to determine then, that part of D and E which belongs to M_2, we seek the region for which

$$\begin{aligned}
&\text{II}(1) < \text{IV}(1) \text{ when } \bar{R}_4(E) \text{ is defined by IV}(1)\\
&\text{II}(1) < \text{IV}(2) \text{ when } \bar{R}_4(E) \text{ is defined by IV}(2)\\
&\text{II}(1) < \text{IV}(3) \text{ when } \bar{R}_4(E) \text{ is defined by IV}(3).
\end{aligned}$$

But within D and E we have that $y^2 < x^2$, so it follows that IV(1) $>$ IV(2) so $\bar{R}_4(E)$ is never defined by IV(2) in D or E. Hence we need determine the points which satisfy the first and third of these relations. Now it is clear that the relation II(1) $<$ IV(1) is equivalent to the relation $|y| < y_0$ for some value y_0 since II(1) is monotonically increasing in y^2 and IV(1) is monotonically decreasing in y^2. Let $y = y_0$ be the line dividing D and E.

We impose a restriction on W_3 such that D is part of M_2 and E is part of M_4.

This restriction is that within E, IV(3) $\leqq$ IV(1), note that since we are concerned only with $|y| < |x|$, this imposes the greatest restriction on W_3 when $x = y = y_0$, so we are requiring that

$$W_3 C e^{-\frac{1}{2}N(y_0^2+y_0^2)} \leqq W_2 C e^{-\frac{1}{2}N y_0^2}$$

or

$$W_3 \leqq W_2 e^{+\frac{1}{2}N y_0^2}.$$

It is simple to see that because of symmetry with respect to both axes and the origin, M_2 is defined by $x^2 + y^2 > r_0^2$ and $|y| < |x|$ and $|y| < y_0$; M_3 by $x^2 + y^2 > r_0^2$ and $|x| < |y|$ and $|x| < x_0$; and M_4 by $x^2 + y^2 > r_0^2$ and $|y| > y_0$ and $|x| > x_0$. It should be pointed out that $x_0 = y_0$.

We now consider the general case with a known covariance matrix. Consider the joint normally distributed variates $X_1^*, X_2^*, \cdots, X_p^*$ whose covariance matrix is $\| \sigma_{ij}^* \|$ $(i, j = 1, 2, \cdots, p)$, where the σ_{ij}^*'s are all known and where $\| \sigma_{ij}^* \|$ is positive definite. The mean values of the X_i^*'s are $\beta_1, \beta_2, \cdots, \beta_p$ which are unknown. It is simple to see that we can consider new variates $X_i = X_i^*/\sqrt{\sigma_{ii}^*}$ whose mean values are $\alpha_i = \beta_i/\sqrt{\sigma_{ii}^*}$ and whose covariance matrix is $\| \sigma_{ij} \|$ where $\sigma_{ii} = 1$. If a sample of N independent observations on the X_i^*'s are given, we have immediately the observations on the X_i's, and we denote the sample means of the X_i's by $x_1, x_2, \cdots, x_p$, respectively.

There are 2^p hypotheses among which we wish to choose; as notation, we let

$$\begin{aligned}
&H_0 \text{ be } \alpha_1 = \alpha_2 = \cdots = \alpha_p = 0\\
&H_1 \text{ be } \alpha_1 \neq 0, \alpha_2 = \alpha_3 = \cdots = \alpha_p = 0\\
&H_2 \text{ be } \alpha_2 \neq 0, \alpha_1 = \alpha_3 = \cdots = \alpha_p = 0\\
&H_{12} \text{ be } \alpha_1\alpha_2 \neq 0, \alpha_3 = \alpha_4 = \cdots = \alpha_p = 0
\end{aligned}$$

etc. As a further abbreviation, let H^1 denote any one of the p hypotheses H_1, $H_2, \cdots, H_p$; let H^2 denote any of the $\binom{p}{2}$ hypotheses $H_{12}, H_{13}, \cdots$; H^3 denote any of the $\binom{p}{3}$ hypotheses $H_{123}, H_{124}, \cdots$; etc. Also let $M_{i_1 i_2 \cdots i_k}$ be the region of the sample space for which we accept the hypothesis $H_{i_1 i_2 \cdots i_k}$, and let $R_{i_1 i_2 \cdots i_k}(\theta, E) = W(\theta, H_{i_1 i_2 \cdots i_k}) f(E \mid \theta)$ be the risk density function if the hypothesis $H_{i_1 i_2 \cdots i_k}$ is chosen, where we have used the notation θ to represent the parameter point $\alpha_1, \alpha_2, \cdots, \alpha_p$.

We will also adopt the following notations: in referring to the parameter point $(\alpha_1, \alpha_2, \cdots, \alpha_p)$, we will write $(i_1, i_2, \cdots, i_k) = 0$ to mean all points for which $\alpha_{i_1} = \alpha_{i_2} = \cdots = \alpha_{i_k} = 0$ and $(\alpha_{j_1})(\alpha_{j_2}) \cdots (\alpha_{j_s}) \neq 0$ where $i_1, i_2, \cdots i_k, j_1, j_2, \cdots, j_s$ are a permutation of the integers $1, 2, \cdots, p$. Furthermore, we will write $[j_1, j_2, \cdots, j_s] \neq 0$ to mean $(i_1, i_2, \cdots, i_k) = 0$.

By Q we denote the covariance matrix of the X_i's and by L its inverse; we will denote the elements of L by λ_{ij}. By $Q^{i_1 i_2 \cdots i_k}$ we denote the matrix obtained by striking out rows $i_1, i_2, \cdots, i_k$ and columns $i_1, i_2, \cdots, i_k$ from Q; by $L^{i_1 i_2 \cdots i_k}$ we denote the inverse of the matrix $Q^{i_1 i_2 \cdots i_k}$, and we will write the elements of

$L^{i_1 i_2 \cdots i_k}$ as $\lambda_{ij}^{i_1 i_2 \cdots i_k}$. Thus we can write the joint distribution of the set of sample means $x_1, x_2, \cdots, x_p$ as

(4) $$Ce^{-\frac{1}{2}N\Sigma\Sigma\lambda_{ij}(x_i-\alpha_i)(x_j-\alpha_j)}.$$

Concerning the definition of the weight function, we will assume the following:

I. If H_0 is accepted,

i) the loss is $W(\Sigma\Sigma\lambda_{ij}\alpha_i\alpha_j)$ if the true parameter point is $(\alpha_1, \alpha_2, \cdots, \alpha_p)$, where W is a continuous, strictly monotonic increasing function whose value is zero if $(1, 2, \cdots, p) = 0$. The function is restricted to increase slowly enough that the product of it and the density function (4) has a finite maximum with respect to the α_i's

II. If H^1 is accepted,

i) consider in particular H_a, then for all parameter points except $(1, 2, \cdots, p) = 0$, the value of the loss is $W(\Sigma\Sigma\lambda_{ij}^a\alpha_i\alpha_j)$, where W is the function defined above.

ii) the loss is W_0^1 if the true parameter point is $(1, 2, \cdots, p) = 0$.

III. If H^2 is accepted,

i) consider in particular H_{ab}, then for all parameter points except $(1, 2, \cdots, p) = 0$ and $[a] \neq 0$ and $[b] \neq 0$, the loss is $W(\Sigma\Sigma\lambda_{ij}^{ab}\alpha_i\alpha_j)$, where W is the function defined above,

ii) the loss is W_1^2 if the true parameter point is either $[a] \neq 0$ or $[b] \neq 0$, where $W_0^1 \leqq W_1^2$,

iii) the loss is W_0^2 if the true parameter point is $(1, 2, \cdots, p) = 0$ where $W_0^2 \geqq W_1^2$.

In general; if H^k is accepted,

i) consider in particular $H_{i_1 i_2 \cdots i_k}$, then for all parameter points except $(1, 2, \cdots, p) = 0$, $[i_1] \neq 0$, $[i_2] \neq 0, \cdots, [i_1, i_2] \neq 0$, $[i_1, i_3] \neq 0, \cdots$, etc., the loss is $W(\Sigma\Sigma\lambda_{ij}^{i_1 i_2 \cdots i_k}\alpha_i\alpha_j)$,

ii) the loss is W_r^k $(r = 1, 2, \cdots, k-1)$ if $[i_{j_1}, i_{j_2}, \cdots, i_{j_r}] \neq 0$, where $j_1, j_2, \cdots, j_r$ are r different positive integers less than or equal to k. Also $W_{k-1}^k \leqq W_{k-2}^k \leqq \cdots \leqq W_1^k$, $W_{k-2}^{k-1} \leqq W_{k-2}^k$, $W_{k-3}^{k-2} \leqq W_{k-3}^{k-1} \leqq W_{k-3}^k$, etc.

iii) the loss is W_0^k if $(1, 2, \cdots, p) = 0$, where $W_1^k \leqq W_0^k$,

where the W_s^t are constants subject to some further slight restrictions which we will impose later. The $\Sigma\Sigma$ has been used throughout to denote summation over all values which i and j take on in $L^{i_1 i_2 \cdots i_k}$.

We consider first the risk density function corresponding to H_0, that is

$$R_0(\theta, E) = W(\Sigma\Sigma\lambda_{ij}\alpha_i\alpha_j)Ce^{-\frac{1}{2}N\Sigma\Sigma\lambda_{ij}(x_i-\alpha_i)(x_j-\alpha_j)}.$$

To maximize $R_0(\theta, E)$, we have the set of p equations obtained by setting the p partials of $R_0(\theta, E)$ with respect to the α_i equal to zero, which are necessary conditions. We have

$$\frac{\partial R_0(\theta, E)}{\partial\alpha_i} = \left\{\frac{\partial W}{\partial\alpha_i} + [N\Sigma\lambda_{ij}(x_j - \alpha_j)]W\right\} Ce^{-\frac{1}{2}N\Sigma\Sigma\lambda_{ij}(x_i-\alpha_i)(x_j-\alpha_j)}$$

so the necessary conditions are

$$\frac{\partial W}{\partial \alpha_i} + [N\Sigma\lambda_{ij}(x_j - \alpha_j)]W = 0 \qquad (i = 1, 2, \cdot \cdot, p)$$

This can also be written

$$(2\Sigma\lambda_{ij}\alpha_j)D_zW(z) + W(z)N\Sigma\lambda_{ij}(x_j - \alpha_j) = 0$$

where we have set $z = \Sigma\Sigma\lambda_{ij}\alpha_i\alpha_j$ and where we use the notation D_z to indicate differentiation with respect to z. Fix i at two particular values, say a and b; then two of the equations of this system can be written

$$(2\Sigma\lambda_{aj}\alpha_j)D_zW(z) + W(z)N\Sigma\lambda_{aj}(x_j - \alpha_j) = 0$$

$$(2\Sigma\lambda_{bj}\alpha_j)D_zW(z) + W(z)N\Sigma\lambda_{bj}(x_j - \alpha_j) = 0$$

that is

$$(\Sigma\lambda_{aj}\alpha_j)[\Sigma\lambda_{bj}(x_j - \alpha_j)] = (\Sigma\lambda_{bj}\alpha_j)[\Sigma\lambda_{aj}(x_j - \alpha_j)]$$

or

$$(\Sigma\lambda_{aj}\alpha_j)(\Sigma\lambda_{bj}x_j) = (\Sigma\lambda_{bj}\alpha_j)(\Sigma\lambda_{aj}x_j).$$

This we can write as

$$\Sigma\Sigma\lambda_{aj}\lambda_{bk}\alpha_jx_k = \Sigma\Sigma\lambda_{bk}\lambda_{aj}\alpha_kx_j$$

or

$$\Sigma\Sigma\lambda_{aj}\lambda_{bk}(\alpha_jx_k - \alpha_kx_j) = 0$$

Giving a and b the p^2 combinations of values which are possible, this is a set of p^2 linear homogeneous equations in the p^2 unknowns $(\alpha_jx_k - \alpha_kx_j)$ which has the obvious solution $\alpha_jx_k - \alpha_kx_j = 0$ or $\alpha_jx_k = \alpha_kx_j$.

Thus we have that the maximum of the function $R_0(\theta, E)$ is reached for a set of values of the α_i's which lie on the straight line

$$\alpha_i = (x_i/x_1)\alpha_1 . \tag{5}$$

The function $\bar{R}_0(E)$, which is the maximum of $R_0(\theta, E)$ with respect to the α_i's is a monotonically increasing function of $(\Sigma\Sigma\lambda_{ij}x_ix_j)$, which we show in the following manner. Because of (5), we see that

$$\begin{aligned}\Sigma\Sigma\lambda_{ij}(x_i - \alpha_i)(x_j - \alpha_j) &= \Sigma\Sigma\lambda_{ij}[x_i - (x_i/x_1)\alpha_1][x_j - (x_j/x_1)\alpha_1] \\ &= \Sigma\Sigma\lambda_{ij}x_ix_j[1 - (\alpha_1/x_1)]^2.\end{aligned}$$

Also,

$$\Sigma\Sigma\lambda_{ij}\alpha_i\alpha_j = \Sigma\Sigma\lambda_{ij}x_ix_j(\alpha_1/x_1)^2.$$

Hence we see that $\bar{R}_0(E)$ is the maximum with respect to ω of

$$W(\omega^2\Sigma\Sigma\lambda_{ij}x_ix_j)Ce^{-\frac{1}{2}N(1-\omega)^2\Sigma\Sigma\lambda_{ij}x_ix_j}$$

so for two sample points $E' = (x'_1, x'_2, \cdots, x'_p)$ and $E'' = (x''_1, x''_2, \cdots, x''_p)$ such that $\Sigma\Sigma\lambda_{ij}x'_ix'_j = \Sigma\Sigma\lambda_{ij}x''_ix''_j$, it is clear that $\bar{R}_0(E') = \bar{R}_0(E'')$; thus $\bar{R}_0(E)$ is a function of $\Sigma\Sigma\lambda_{ij}x_ix_j$.

But then without loss of generality, we can consider $\bar{R}_0(E)$ along the x_1 axis, i.e for $x_2 = x_3 = \cdots = x_p = 0$. Using relation (5), we see that this implies that the maximizing parameter values are $\alpha_2 = \alpha_3 = \cdots = \alpha_p = 0$. But then

$$\bar{R}_0(E) = \max_{\alpha_1}. \text{ of } W(\lambda_{11}\alpha_1^2)Ce^{-\frac{1}{2}N\lambda_{11}(x_1-\alpha_1)^2}$$

which we have previously shown is a monotonic increasing function of x_1^2. Therefore $\bar{R}_0(E)$ is a monotonic increasing function of $\Sigma\Sigma\lambda_{ij}x_ix_j$.

We will furthermore show that the maximum of each risk density function corresponding to parts i) as given in the weight functions are monotonically increasing functions of certain quadratic forms in the x_i. Consider for example the function corresponding to part i) of $R_1(\theta, E)$, that is

$$W(\Sigma\Sigma\lambda_{ij}^1\alpha_i\alpha_j)Ce^{-\frac{1}{2}N\Sigma\Sigma\lambda_{ij}(x_i-\alpha_i)(x_j-\alpha_j)}. \tag{6}$$

We will write the maximum of this function with respect to the α_i's as $\bar{R}_1(i)$. Note that the weight function is not a function of α_1, hence the partial derivative of (6) with respect to α_1 set equal to zero is equivalent to

$$\Sigma\lambda_{1j}(x_j - \alpha_j) = 0.$$

Squaring this relation and multiplying by $N/2\lambda_{11}$ gives

$$(N/2\lambda_{11})\Sigma\Sigma\lambda_{1i}\lambda_{1j}(x_i - \alpha_i)(x_j - \alpha_j) = 0$$

so we can write the exponent in (6)

$$\text{Exp.} = -(N/2\lambda_{11})\Sigma\Sigma(\lambda_{11}\lambda_{ij} - \lambda_{1i}\lambda_{1j})(x_i - \alpha_i)(x_j - \alpha_j).$$

Because of the definition of λ_{ij}, if we write ω_{ij} for the cofactor of σ_{ij} in $|\sigma_{ij}|$, we have

$$\text{Exp.} = -[N/2\lambda_{11}(|\sigma_{ij}|)^2]\Sigma\Sigma(\omega_{11}\omega_{ij} - \omega_{1i}\omega_{1j})(x_i - \alpha_i)(x_j - \alpha_j).$$

But by a well known algebraic identity[2],

$$\begin{aligned}\omega_{11}\omega_{ij} - \omega_{1i}\omega_{1j} &= |\sigma_{ij}| \cdot [\text{cofactor of } (\sigma_{11}\sigma_{ij} - \sigma_{1i}\sigma_{1j}) \text{ in } |\sigma_{ij}|] \\ &= |\sigma_{ij}| \cdot \omega_{ij}^1\end{aligned}$$

where we have written ω_{ij}^1 to be the cofactor of σ_{ij} in $|\sigma_{ij}^1|$, so

$$\text{Exp.} = -(N/2\lambda_{11}|\sigma_{ij}|)\Sigma\Sigma\omega_{ij}^1(x_i - \alpha_i)(x_j - \alpha_j).$$

But $\lambda_{11}|\sigma_{ij}| = \omega_{11} = |\sigma_{ij}^1|$, hence

$$\text{Exp.} = -\frac{N}{2}\Sigma\Sigma\lambda_{ij}^1(x_i - \alpha_i)(x_j - \alpha_j).$$

Therefore

$$\bar{R}_1(i) = \max_{\text{all } \alpha_i\text{'s}}. \text{ of } W(\Sigma\Sigma\lambda_{ij}^1\alpha_i\alpha_j)Ce^{-\frac{1}{2}N\Sigma\Sigma\lambda_{ij}^1(x_i-\alpha_i)(x_j-\alpha_j)}$$

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[2] See M Bocher, *Introduction to Higher Algebra*

But then it follows in exactly the same way as with $\bar{R}_0(E)$ that $\bar{R}_1(i)$ is a monotonically increasing function of $\Sigma\Sigma\lambda^1_{ij}x_ix_j$. For the other functions $\bar{R}_k(i)$ corresponding to other hypotheses H^1, the argument is identical, and for risk density functions corresponding to hypotheses with more than one $\alpha_i \neq 0$, the same argument is repeated two or more times in succession to give the result.

We will show that for any value of the parameters $\alpha_1, \alpha_2, \cdots, \alpha_p$ the relation

$$\Sigma\Sigma\lambda_{ij}\alpha_i\alpha_j \geqq \Sigma\Sigma\lambda^1_{ij}\alpha_i\alpha_j$$

holds. This relation is true if the relation

$$\Sigma\Sigma[(\omega_{ij}/\,|\,\sigma_{ij}\,|) - (\omega^1_{ij}/\,|\,\sigma^1_{ij}\,|)]\alpha_i\alpha_j \geqq 0 \tag{7}$$

is true where we define $\omega^1_{1j} = 0$. That is, if

$$(1/\,|\,\sigma_{ij}\,|\,|\,\sigma^1_{ij}\,|)\Sigma\Sigma\omega_{ij}\omega_{11} - \omega^1_{ij}\,|\,\sigma_{ij}\,|)\alpha_i\alpha_j \geqq 0$$

where we have substituted ω_{11} for its equal $|\,\sigma^1_{ij}\,|$. But note that

$$\omega^1_{ij} = \text{cofactor of } (\sigma_{11}\sigma_{ij} - \sigma_{1i}\sigma_{1j}) \text{ in } |\,\sigma_{ij}\,|$$

hence by the identity quoted (see footnote 2)

$$|\,\sigma_{ij}\,|\,\omega^1_{ij} = \omega_{11}\omega_{ij} - \omega_{1i}\omega_{1j}$$

so the left hand member of relation (7) is

$$\begin{aligned}(1/|\,\sigma_{ij}\,||\,\sigma^1_{ij}\,|)\Sigma\Sigma(\omega_{ij}\omega_{11} - \omega_{11}\omega_{ij} + \omega_{1i}\omega_{1j})\alpha_i\alpha_j \\ = (1/|\,\sigma_{ij}\,||\,\sigma^1_{ij}\,|)\Sigma\Sigma\omega_{1i}\omega_{1j}\alpha_i\alpha_j \\ = [\Sigma\omega_{1i}\alpha_i]^2/(|\,\sigma_{ij}\,||\,\sigma^1_{ij}\,|) \\ \geqq 0\end{aligned}$$

since all matrices here are symmetric and positive definite. Note that the argument can be repeated one or more times to show

$$W(\Sigma\Sigma\lambda_{ij}\alpha_i\alpha_j) \geqq W(\Sigma\Sigma\lambda^{i_1i_2\cdots i_k}_{ij}\alpha_i\alpha_j)$$

or

$$W(\Sigma\Sigma\lambda^{j_1j_2\cdots j_s}_{ij}\alpha_i\alpha_j) \geqq W(\Sigma\Sigma\lambda^{i_1i_2\cdots i_k}_{ij}\alpha_i\alpha_j)$$

where $i_1i_2, \cdots, i_k$ are any set of k different integers less than or equal to p, and $j_1j_2 \cdots, j_s$ are any subset of $i_1i_2 \cdots, i_k$.

Consider the maximum of the expressions

$$W^k_rCe^{-\frac{1}{2}N\Sigma\Sigma\lambda_{ij}(x_i-\alpha_i)(x_j-\alpha_j)}.$$

We know that $(p - r)$ of the α_i's in these expressions are zero and by an argument similar to that given above[3], it is clear that if the r α_i's not equal to zero are $\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_r}$, then the maximum of the expressions is given by

$$W^k_rCe^{-\frac{1}{2}N\Sigma\Sigma\lambda^{i_1i_2\cdots i_r}_{ij}x_ix_j}.$$

[3] See p. 36.

Also for $r = 0$, the maximum is obviously

$$W_0^k C e^{-\frac{1}{2}N\Sigma\Sigma\lambda_{ij}x_ix_j}.$$

Recall that we have restricted the W_r^k's so that

$$(8) \qquad W_0^1 \leqq W_0^2 \leqq \cdots \leqq W_0^p \quad \text{and} \quad W_{p-1}^p \leqq \cdots \leqq W_0^p .$$

From a previous calculation, it follows that

$$(9) \qquad \Sigma\Sigma\lambda_{ij}x_ix_j \geqq \Sigma\Sigma\lambda_{ij}^{i_1}x_ix_j \geqq \Sigma\Sigma\lambda_{ij}^{i_1i_2}x_ix_j \geqq \cdots$$

We can then quite easily calculate the region M_0, that is, the region of the sample space for which $\bar{R}_0(E)$ is the minimum of all the $\bar{R}_{i_1i_2\cdots i_k}(E)$'s. We have pointed out that

$$W(\Sigma\Sigma\lambda_{ij}\alpha_i\alpha_j) \geqq W(\Sigma\Sigma\lambda_{ij}^{i_1i_2\cdots i_k}\alpha_i\alpha_j)$$

so it follows that

$$(10) \qquad \bar{R}_0(E) \geqq \bar{R}_{i_1i_2\cdots i_k}(i)$$

that is

$$\bar{R}_0(E) \geqq \bar{R}_{i_1i_2\cdots i_k}(E)$$

so long as $\bar{R}_{i_1i_2\cdots i_k}(E)$ is defined by $\bar{R}_{i_1i_2\cdots i_k}(i)$.

From the relations (8) and (9), we have that

$$(11) \qquad W_0^1 C e^{-\frac{1}{2}N\Sigma\Sigma\lambda_{ij}x_ix_j} \leqq W_0^k C e^{-\frac{1}{2}N\Sigma\Sigma\lambda_{ij}x_ix_j}$$

for $k = 2, 3, \cdots, p$. Now because

$$W_0^1 C e^{-\frac{1}{2}N\Sigma\Sigma\lambda_{ij}x_ix_j}$$

is a monotonic decreasing function of $\Sigma\Sigma\lambda_{ij}x_ix_j$, and because $\bar{R}_0(E)$ is a monotonically increasing function of $\Sigma\Sigma\lambda_{ij}x_ix_j$, there is a value r_0^2 such that within the ellipse $\Sigma\Sigma\lambda_{ij}x_ix_j = r_0^2$, the relation

$$(12) \qquad \bar{R}_0(E) < W_0^1 C e^{-\frac{1}{2}N\Sigma\Sigma\lambda_{ij}x_ix_j}$$

holds, and outside it the opposite inequality holds. But from relations (10) and (12), it follows that within this ellipse, no $\bar{R}_{i_1i_2\cdots i_k}(E)$ except $\bar{R}_0(E)$ can be defined by $\bar{R}_{i_1i_2\cdots i_k}(i)$. Then in view of relation (11) and since a quantity is certainly less than the maximum of several quantities if it is less than one of those several quantities, the region M_0 is the set of points $\Sigma\Sigma\lambda_{ij}x_ix_j < r_0^2$.

Now consider the functions $\bar{R}_a(E)$ in the region outside M_0. We know that $\bar{R}_a(E) = \bar{R}_a(i)$ when

$$\max_{\alpha_i\text{'s}} \text{ of } W(\Sigma\Sigma\lambda_{ij}^a\alpha_i\alpha_j)e^{-\frac{1}{2}N\Sigma\Sigma\lambda_{ij}^a(x_i-\alpha_i)(x_j-\alpha_j)} \geqq W_0^1 e^{-\frac{1}{2}N\Sigma\Sigma\lambda_{ij}x_ix_j}$$

and we will write $\bar{R}_i(E) = \bar{R}_i(ii)$ when the opposite inequality holds. Consider a part of the sample space outside M_0 in which

$$\begin{aligned}\bar{R}_{i_1}(E) &= \bar{R}_{i_1}(ii)\\ \bar{R}_{i_2}(E) &= \bar{R}_{i_2}(ii)\\ \cdots\cdots\cdots &\cdots\\ \bar{R}_{i_k}(E) &= \bar{R}_{i_k}(ii)\end{aligned}$$

where $k \geqq 1$, and where $\bar{R}_j(E) \neq \bar{R}_j(ii)$ for $j \neq i_1, i_2, \cdots, i_k$. We see in this case that $\bar{R}_{i_1}(E) = \bar{R}_{i_2}(E) = \cdots = \bar{R}_{i_k}(E) < \bar{R}_j(E)$, where again $j \neq i_1, i_2, \cdots, i_k$. Furthermore, in this case, because of the relation (11), we have that E should be a point of either $M_{i_1}, M_{i_2}, \cdots$ or, M_{i_k}. We will arbitrarily decide in this case that E should be a point of M_{i_s} (s an integer $\leqq k$) where i_s is determined so that

$$\Sigma\Sigma\lambda_{ij}^{i_s}x_ix_j \leqq \Sigma\Sigma\lambda_{ij}^{i_t}x_ix_j \qquad \text{for any } t = 1, 2, \cdots, k.$$

Now consider the region in which $\bar{R}_r(E) = \bar{R}_r(i)$ for all $r = 1, 2, \cdots, p$. We see that each $\bar{R}_r(i)$ is the same monotonically increasing function of a quadratic form of the type $\Sigma\Sigma\lambda_{ij}^r x_ix_j$. Hence in order that E be a point of a particular M_r, it is necessary that

$$\Sigma\Sigma\lambda_{ij}^r x_ix_j \leqq \Sigma\Sigma\lambda_{ij}^s x_ix_j \qquad \text{for all } s \neq r \tag{13}$$

Now let us consider a fixed r and compare $\bar{R}_r(i)$ with all $\bar{R}_{ri_1i_2\cdots i_k}(E)$'s for $k \geqq 1$. We have pointed out that

$$\Sigma\Sigma\lambda_{ij}^r x_ix_j \geqq \Sigma\Sigma\lambda_{ij}^{ri_1i_2\cdots i_k}x_ix_j \tag{14}$$

so $\bar{R}_r(i) \geqq \bar{R}_{ri_1i_2\cdots i_k}(i)$ and hence $\bar{R}_r(i)$ can be a minimum at most when all $\bar{R}_{ri_1i_2\cdots i_k}(E)$'s are defined by other than $\bar{R}_{ri_1i_2\cdots i_k}(i)$.

Consider then, any $\bar{R}_{ri_1}(E)$ when defined by other than $\bar{R}_{ri_1}(i)$, that is when $\bar{R}_{ri_1}(E)$ is equal to one of

$$\begin{aligned}W_1^2Ce^{-\frac{1}{2}N\Sigma\Sigma\lambda_{ij}^r x_ix_j} &= \bar{R}_{ri_1}(ii) \quad \text{(say)}\\ W_1^2Ce^{-\frac{1}{2}N\Sigma\Sigma\lambda_{ij}^{i_1} x_ix_j} &= \bar{R}_{ri_1}(iii) \quad \text{(say)}\\ W_0^2Ce^{-\frac{1}{2}N\Sigma\Sigma\lambda_{ij} x_ix_j} &= \bar{R}_{ri_1}(iv) \quad \text{(say)}\end{aligned}$$

Because of the relations (8) and (14), we have that

$$\bar{R}_{ri_1}(E) \leqq \bar{R}_{ri_1i_2\cdots i_k}(E)$$

whenever these are defined by other than $\bar{R}_{ri_1}(i)$ and $\bar{R}_{ri_1i_2\cdots i_k}(i)$. Furthermore in the region defined by (13), we see that $\bar{R}_{ri_1}(ii) \geqq \bar{R}_{ri_1}(iii)$, hence $\bar{R}_{ri_1}(E)$ is never defined by $\bar{R}_{ri_1}(iii)$ in this region.

Now the relation $\bar{R}_r(i) < \bar{R}_{ri_1}(ii)$ is easily seen to be equivalent to the relation

$$\Sigma\Sigma\lambda_{ij}^r x_ix_j < r_1^2 \tag{15}$$

for some value r_1. With the restriction on W_0^2 that it be not so much larger than W_1^2 that when (12) does not hold, $\bar{R}_{ri_1}(E)$ is not defined by $\bar{R}_{ri_1}(iv)$, we have that the region for which $\bar{R}_r(i) < \bar{R}_{ri_1i_2\cdots i_k}(E)$ is the region defined by (13) and (15).

We then restrict the relationship between the constants W_0^1 and W_0^2 to be such that for all points outside of M_0 but within the region defined by (13) and (15), the relation $\Sigma\Sigma\lambda_{ij}^{j_1j_2\cdots j_k}x_ix_j \geqq \Sigma\Sigma\lambda_{ij}^r x_ix_j$ holds for $j_1, j_2, \cdots, j_k$ each different from r. Note that this is not an unreasonable restriction since the right hand side of the relation is bounded above by r_1^2, $\Sigma\Sigma\lambda_{ij}x_ix_j$ is bounded below by r_0^2, and therefore, $\Sigma\Sigma\lambda_{ij}^{j_1j_2\cdots j_k}x_ix_j$ is bounded below by some positive value r^2 where r^2 is a monotonically increasing function of r_0^2

Using a similar method, the region $M_{i_1i_2\cdots i_k}$ can be obtained after all regions $M_{i_1i_2\cdots i_m}$ for all $m < k$ have been derived. If some further restrictions are imposed on the constants in the weight functions similar to those formulated in deriving the region M_r, it can be shown that the region $M_{i_1i_2\cdots i_k}(k \geqq 1)$ will be given by the inequalities

$$\Sigma\Sigma\lambda_{ij}x_ix_j \geqq r_0^2$$

$$\Sigma\Sigma\lambda_{ij}^{j_1j_2\cdots j_m}x_ix_j \geqq r_m^2 \qquad \text{for all } m < k \text{ and all } j_1, \cdots, j_m$$

$$\Sigma\Sigma\lambda_{ij}^{i_1i_2\cdots i_k}x_ix_j \leqq \Sigma\Sigma\lambda_{ij}^{j_1j_2\cdots j_k}x_ix_j \qquad \text{for all } j_1, \cdots, j_k$$

and

$$\Sigma\Sigma\lambda_{ij}^{i_1i_2\cdots i_k}x_ix_j < r_k^2$$

Thus we have rationalized the following solution of the question posed at the beginning of section 4. We test the hypothesis $E(x_1) = E(x_2) = \cdots = E(x_p) = 0$ using the generalized Student ratio replacing the sample covariance matrix by the population covariance matrix since the latter is assumed to be known, at some chosen level of significance. If the hypothesis is not rejected, we make the decision corresponding to H_0. If the ratio is significant, we compute the ratios $T^1, T^2, \cdots, T^p$ where by definition $T^{i_1i_2\cdots i_k}$ is the generalized Student ratio computed for $x_{j_1}, x_{j_2}, \cdots, x_{j_s}$ ($i_1, i_2, \cdots, i_k, j_1, j_2, \cdots, j_s$ is a permutation of the integers $1, 2, \cdots, p$), the variates $x_{i_1}, x_{i_2}, \cdots, x_{i_k}$ being ignored.

We consider the smallest of the ratios computed on the basis of $(p - 1)$ of the x_i's; say it is T^r. Then if T^r is not significant at some level of significance (which need not be the same level as considered before), we make the decision corresponding to H_r; if T^r is significant, we compute all the ratios based on $(p - 2)$ of the x's. If T^{rs} is the smallest of these, we make the decision corresponding to H_{rs} if T^{rs} is not significant but proceed to calculate the ratios based on $(p - 3)$ of the x_i's if it is significant, and so on.

5. Concluding remarks. It should be pointed out that while the derivation of the explicit inequalities defining the various regions of acceptance may be

rather involved, for any given sample point E, it is relatively simple to determine the region of acceptance to which this point E belongs. That is, we calculate the various values $\bar{R}_{i_1 i_2 \cdots i_k}(E)$ and choose the decision $H_{j_1 \cdots j_k}$ if $\bar{R}_{j_1 j_2 \cdots j_k}(E)$ is the minimum of the values of $\bar{R}_{i_1 i_2 \cdots i_k}(E)$ for all values of $i_1, i_2, \cdots, i_k$. For making a decision on the basis of a given sample point E, it is not necessary to find explicit analytic formulas defining the shapes of the various regions of acceptance.

Since the principle used here is proposed merely as a substitute for Wald's principle for the sake of mathematical simplification, it is felt that in certain problems Wald's principle may be used as a check on the results. For example, it is felt that the new principle is apt to lead to decision regions of the proper shape though the exact sizes of these regions may not be correct. In cases where the decision regions cannot be determined by Wald's principle, it seems possible that a determination may be made in Wald's sense among the various decision regions having the same shapes as those given by the new principle. In the case considered here, for example, it may be possible to determine new values of $\tau_0^2, \tau_1^2, \cdots, \tau_{p-1}^2$.

I should like to express my very great appreciation to Professor H. Hotelling for many suggestions during the preparation of this paper and to Professor A. Wald for constant guidance. I should also like to credit Professor Helen Walker with originally posing the question that led to this research.

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A TWO-SAMPLE TEST FOR A LINEAR HYPOTHESIS WHOSE POWER IS INDEPENDENT OF THE VARIANCE

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1. Introduction. In a paper in the *Annals of Mathematical Statistics*, Dantzig [1] proves that, for a sample of fixed size, there does not exist a test for Student's hypothesis whose power is independent of the variance. Here, a two-sample test with this property will be presented, the size of the second sample depending upon the result of the first. The problem of determining confidence intervals, of preassigned length and confidence coefficient, for the mean of a normal distribution with unknown variance is solved by the same procedure. These considerations including the non-existence of a single-sample test whose power is independent of the variance, are extended to the case of a linear hypothesis. In order to make the power of a test or the length of a confidence interval exactly independent of the variance, it appears necessary to waste a small part of the information. Thus, in practical applications, one will not use a test with this property, but rather a test which is uniformly more powerful, or an interval of the same length, whose confidence coefficient is a function of σ, but always greater than the desired value, the difference usually being slight, at the same time reducing the expected number of observations by a small amount.

Any two sample procedure, such as that discussed in this paper, can be considered a special case of sequential analysis developed by Wald [5].

The problem of whether these tests and confidence intervals are in any sense optimum is unsolved. It is difficult even to formulate a definition of an optimum among sequential tests of a hypothesis against multiple alternatives. However it is shown that, if the variance and initial sample size are sufficiently large, the expected number of observations differs only slightly from the number of observations required for a single-sample test when the variance is known. It also seems likely that the confidence intervals do possess some optimum property among the class of all two-sample procedures.

Although Student's hypothesis is a special case of a linear hypothesis, it is treated separately, because it illustrates the basic idea without any complicated notation or new distributions. The test for Student's hypothesis involves the use only of Student's distribution, even for the power of the test, while the power function of the test proposed here for a linear hypothesis involves a new type of non-central F-distribution.

The notation χ^2_n is used as a generic symbol for a random variable equal to the sum of squares of n independently normally distributed random variables with mean 0 and variance 1, i.e., χ^2_n has the χ^2 distribution with n degrees of freedom,

$$P\{\chi_n^2 < T\} = \frac{1}{(\sqrt{2})^n\Gamma(\frac{1}{2}n)}\int_0^T e^{-\frac{1}{2}u}u^{\frac{1}{2}n-1}\,du \qquad \text{for } T \geq 0$$

$$= 0 \qquad \text{for } T \leq 0.$$

The notation t_n is used as a generic symbol for $\frac{x\sqrt{n}}{\chi_n}$, where x is normally distributed with mean 0 and variance 1, independently of χ_n^2, i.e., t_n has the distribution of Student's t with n degrees of freedom,

$$P\{t_n < t\} = \frac{\Gamma(\frac{1}{2}(n+1))}{\sqrt{n\pi}\,\Gamma(\frac{1}{2}n)}\int_{-\infty}^t \left(1 + \frac{z^2}{n}\right)^{-\frac{1}{2}(n+1)} dz.$$

$F_{m,n}$ is a generic symbol for a random variable of the form $F_{m,n} = n\chi_m^2/m\chi_n^2$, the numerator and denominator being independently distributed, i.e., $F_{m,n}$ has the distribution of an F-ratio with m and n degrees of freedom,

$$P\{F_{m,n} < T\} = \frac{\Gamma(\frac{1}{2}(m+n))}{\Gamma(\frac{1}{2}m)\Gamma(\frac{1}{2}n)}\int_0^T \left(\frac{m}{n}\right)^{\frac{1}{2}m} F^{\frac{1}{2}m-1}\left(1 + \frac{m}{n}F\right)^{-\frac{1}{2}(m+n)} dF.$$

A symbol of the above type with an additional subscript α denotes the upper $100\alpha\%$ significance level, e.g., $t_{n,\alpha}$ is defined by

$$P\{t_n > t_{n,\alpha}\} = \alpha.$$

The symbol $E\{x \mid Q(x)\}$ denotes the set of all x such that the condition $Q(x)$ holds. This should not be confused with $E(x \mid T)$, which denotes the expected value of a random variable x, given the conditions T.

The size of a critical region is the probability that the sample point will lie within the region under the null hypothesis. The terms length and volume, as applied to confidence regions are used in the ordinary geometrical sense.

2. The test for Student's hypothesis. Suppose x_i, $i = 1, 2, \cdots$ are independently normally distributed with mean ξ and variance σ^2. We wish to test the hypothesis $\xi = \xi_0$, the power of the test to depend only upon $\xi - \xi_0$, not upon σ^2. For this purpose we define a statistic t' as follows. A sample of n_0 observations, $x_1 \cdots x_{n_0}$ is taken, and the sample estimate, s^2, of the variance computed by

$$s^2 = \frac{1}{n_0 - 1}\left\{\sum_1^{n_0} x_i^2 - \frac{1}{n_0}\left(\sum_1^{n_0} x_i\right)^2\right\}. \tag{1}$$

Then n is defined by

$$n = \max\left\{\left[\frac{s^2}{z}\right] + 1, n_0 + 1\right\}, \tag{2}$$

where z is a previously specified positive constant, $[q]$ denoting the smallest integer less than q. Additional observations, $x_{n_0+1}, \cdots, x_n$ are taken, and, in

accordance with an initially specified rule depending only upon s^2, real numbers a_i, $i = 1 \cdots n$ are chosen in such a way that

$$(3) \qquad \sum_1^n a_i = 1, \qquad a_1 = a_2 = \cdots = a_{n_0}$$
$$s^2 \sum_1^n a_i^2 = z.$$

This is clearly possible since

$$(4) \qquad \min \sum_1^n a_i^2 = \frac{1}{n} \leq \frac{z}{s^2} \quad \text{by (2)},$$

the minimum being taken subject to the conditions

$$\sum_1^n a_i = 1, \qquad a_1 = a_2 = \cdots = a_{n_0}.$$

Then t' is defined by

$$(5) \qquad t' = \frac{\sum_1^n a_i x_i - \xi_0}{\sqrt{z}} = \frac{\sum_1^n a_i(x_i - \xi)}{\sqrt{z}} + \frac{\xi - \xi_0}{\sqrt{z}}$$
$$= u + \frac{\xi - \xi_0}{\sqrt{z}},$$

where

$$(6) \qquad u = \frac{\sum_1^n a_i(x_i - \xi)}{\sqrt{z}}.$$

Then u has the distribution of Student's t with $n_0 - 1$ degrees of freedom, regardless of the value of σ^2. For $(n_0 - 1)s^2/\sigma^2$ has the distribution of $\chi^2_{n_0-1}$ and the conditional distribution of $\frac{1}{\sqrt{z}} \sum_1^n a_i(x_i - \xi) = u$, given s, is normal with mean 0 and variance $\sigma^2 \Sigma a_i^2/z = \sigma^2/s^2$. But the usual form of a random variable t_{n_0-1} is $t_{n_0-1} = y/s$, y being normally distributed with mean 0 and variance σ^2, and $(n_0 - 1)s^2/\sigma^2$ having the distribution of $\chi^2_{n_0-1}$, independent of y. Thus the conditional distribution of u, given s, is normal with mean 0 and variance σ^2/s^2, so that t_{n_0-1} and u have the same distribution.

This theorem can be used to obtain an unbiased test for the hypothesis H_0 that $\xi = \xi_0$, the power being independent of σ^2, which is supposed unknown. Let α be the desired size of the critical region and let $t_{n_0-1,\alpha/2}$ be such that

$$(7) \qquad P\{t_{n_0-1} > t_{n_0-1,\alpha/2}\} = \frac{\alpha}{2}.$$

Then if we reject H_0 whenever

$$\left|\frac{\sum_1^n a_i x_i - \xi_0}{\sqrt{z}}\right| > t_{n_0-1,\alpha/2}, \tag{8}$$

we obtain an unbiased test of H_0, whose power function is $1 - \beta(\xi)$ where

$$\beta(\xi) = P\left\{-t_{n_0-1,\alpha/2} + \frac{\xi_0 - \xi}{\sqrt{z}} < t_{n_0-1} < t_{n_0-1,\alpha/2} + \frac{\xi_0 - \xi}{\sqrt{z}}\right\}. \tag{9}$$

The fact that the test is unbiased follows immediately from the symmetry and unimodality of the t distribution.

If we wish to test the hypothesis $H_0: \xi = \xi_0$ against one-sided alternatives $\xi > \xi_0$, the procedure is similar. The critical region of size α is defined by

$$\frac{\sum_1^n a_i x_i - \xi_0}{\sqrt{z}} > t_{n_0-1,\alpha} \tag{10}$$

and the power function is

$$1 - \beta(\xi) = P\left\{t_{n_0-1} > t_{n_0-1,\alpha} + \frac{\xi_0 - \xi}{\sqrt{z}}\right\}. \tag{11}$$

A confidence interval for ξ, of predetermined length l and confidence coefficient $1 - \alpha$ can be obtained by selecting z so that

$$\begin{aligned}
1 - \alpha &= P\left\{-\frac{l}{2\sqrt{z}} < t_{n_0-1} < \frac{l}{2\sqrt{z}}\right\} \\
&= P\left\{-\frac{l}{2\sqrt{z}} < \frac{\sum_1^n a_i(x_i - \xi)}{\sqrt{z}} < \frac{l}{2\sqrt{z}}\right\} \\
&= P\left\{\xi - \frac{l}{2} < \sum_1^n a_i x_i < \xi + \frac{l}{2}\right\} \\
&= P\left\{\left|\sum_1^n a_i x_i - \xi\right| < \frac{l}{2}\right\} \\
&= P\left\{\sum_1^n a_i x_i - \frac{l}{2} < \xi < \sum_1^n a_i x_i + \frac{l}{2}\right\},
\end{aligned} \tag{12}$$

where ξ is the true mean of the distribution. Thus $(\Sigma a_i x_i - l/2, \Sigma a_i x_i + l/2)$ is the desired confidence interval.

In the above tests and confidence intervals, the distribution of the required number of observations, n, is

$$P\{n = n_0 + 1\} = P\left\{\frac{s^2}{z} \leq n_0 + 1\right\}$$

$$
\begin{aligned}
(13)\qquad &= P\{(n_0 - 1)s^2/\sigma^2 < (n_0 + 1)(n_0 - 1)z/\sigma^2\} = P\{\chi^2_{n_0-1} < y\} \\
&= \frac{1}{(\sqrt{2})^{n_0-1}\Gamma(\frac{1}{2}(n_0 - 1))}\int_0^y e^{-\frac{1}{2}u}u^{\frac{1}{2}(n_0-3)}\,du,
\end{aligned}
$$

where $y = (n_0^2 - 1)\, z/\sigma^2$,

$$
\begin{aligned}
P\{n = \nu\} &= P\left\{\nu < \frac{s^2}{z} + 1 \leq \nu + 1\right\} \\
(14)\qquad &= P\{(\nu - 1)(n_0 - 1)z/\sigma^2 < \chi^2_{n_0-1} < \nu(n_0 - 1)z/\sigma^2\} \\
&= \frac{1}{(\sqrt{2})^{n_0-1}\Gamma(\frac{1}{2}(n_0 - 1))}\int_{(\nu-1)(n_0-1)z/\sigma^2}^{\nu(n_0-1)z/\sigma^2} e^{-\frac{1}{2}u}u^{\frac{1}{2}(n_0-3)}\,du,
\end{aligned}
$$

for integral $\nu > n_0 + 1$, all other values being impossible. Thus the expected number of observations, $E(n)$, satisfies the inequalities

$$
\begin{aligned}
&\frac{1}{(\sqrt{2})^{n_0-1}\Gamma(\frac{1}{2}(n_0 - 1))}\left\{\int_0^y (n_0 + 1)e^{-\frac{1}{2}u}u^{\frac{1}{2}(n_0-3)} + \int_y^\infty e^{-\frac{1}{2}u}u^{\frac{1}{2}(n_0-3)}\frac{\sigma^2 u}{z(n_0 - 1)}\,du\right\} \\
&\quad < E(n) \\
(15)\qquad &\quad < \frac{1}{(\sqrt{2})^{n_0-1}\Gamma(\frac{1}{2}(n_0 - 1))}\left\{\int_0^y (n_0 + 1)e^{-\frac{1}{2}u}u^{\frac{1}{2}(n_0-3)}\,du + \int_y^\infty e^{-\frac{1}{2}u}u^{\frac{1}{2}(n_0-3)}\right. \\
&\qquad\qquad \left.\cdot\left(\frac{\sigma^2 u}{z(n_0 - 1)} + 1\right)du\right\},
\end{aligned}
$$

which can be rewritten

$$
\begin{aligned}
&(n_0 + 1)P\{\chi^2_{n_0-1} < y\} + \frac{\sigma^2}{z}P\{\chi^2_{n_0+1} > y\} \\
(16)\qquad &< E(n) < (n_0 + 1)P\{\chi^2_{n_0-1} < y\} + \frac{\sigma^2}{z}P\{\chi^2_{n_0+1} > y\} + P\{\chi^2_{n_0-1} > y\}.
\end{aligned}
$$

Consequently $E(n)$ is a function of σ^2, and can be evaluated from tables of the incomplete Γ function.

As mentioned in the introduction, these tests and confidence intervals will not be used exactly in this form, since they waste information in order to make the power of the test or the length of the confidence interval strictly independent of the variance. Instead of (2) we take a total of

$$
(17)\qquad n = \max\left\{\left[\frac{s^2}{z}\right] + 1, n_0\right\}
$$

observations, and define

$$
\begin{aligned}
t'' &= \frac{\left(\frac{1}{n}\sum_1^n x_i - \xi_0\right)\sqrt{n}}{s} \\
(18)\qquad &= \frac{\frac{1}{n}\sum_1^n (x_i - \xi)}{s}\sqrt{n} + \frac{\xi - \xi_0}{s}\sqrt{n} \\
&= u' + \frac{\xi - \xi_0}{s}\sqrt{n}.
\end{aligned}
$$

By the same reasoning as that following (6), u' has the t distribution with $n_0 - 1$ degrees of freedom By (2)

$$n \geq s^2/z \quad \text{so that, although} \quad \left|\frac{\xi - \xi_0}{s}\sqrt{n}\right| \tag{19}$$

is a random variable,

$$\left|\frac{\xi - \xi_0}{s}\sqrt{n}\right| \geq \left|\frac{\xi - \xi_0}{\sqrt{z}}\right| \tag{20}$$

Thus, if we use

$$|t''| > t_{n_0-1,\alpha/2} \text{ or } t'' > t_{n_0-1,\alpha} \tag{21}$$

instead of (8) or (10) respectively, we shall always increase the power of the test. Also the expected number of observations will be reduced from that in (16) by $P\{\chi^2_{n_0-1} < y\}$. Similarly if z is defined as in (12), the interval

$$\left(\frac{1}{n}\sum_1^n x_i - \frac{l}{2}, \qquad \frac{1}{n}\sum_1^n x_i + \frac{l}{2}\right)$$

has length l, and the probability that it covers the true mean ξ is a function of σ, but is always greater than $1 - \alpha$, and differs only slightly from $1 - \alpha$ if $\sigma^2 > n_0 z$. Thus it can be used instead of the confidence interval (12).

From (16) it follows that

$$\overline{\lim_{\sigma\to\infty}}\left\{E(n) - \frac{\sigma^2}{z}\right\} \leq 1$$

$$\underline{\lim_{\sigma\to\infty}}\left\{E(n) - \frac{\sigma^2}{z}\right\} \geq 0,$$

the approximation $E(n) \approx \sigma^2/z$ being fair provided $\sigma^2 > zn_0$. The length of the confidence interval (12) is given by

$$l = 2t_{n_0-1,\alpha/2}\sqrt{z} \approx \frac{2\sigma t_{n_0-1,\alpha/2}}{\sqrt{E(n)}}$$

When the variance σ^2 is known, the length of the single-sample confidence interval of confidence coefficient $1 - \alpha$ obtained on the basis of n observations is given by

$$1 - \alpha = \frac{1}{\sqrt{2\pi}}\int_{-l\sqrt{n}/2\sigma}^{l\sqrt{n}/2\sigma} e^{-\frac{1}{2}x^2}\,dx$$

i.e.,

$$l = 2t_{\infty,\alpha/2}\sigma/\sqrt{n}\,.$$

Since, even for moderate values of n_0, say $n_0 \geq 30$, $t_{n_0-1,\alpha/2}$ differs only slightly from $t_{\infty,\alpha/2}$, the expected number of observations for a confidence interval of

given length and confidence coefficient is only slightly larger than the fixed number of observations required in the single-sample case when the variance is known provided the variance is moderately large.

3. Distribution of a non-central F-ratio. In the extention of the above considerations to the testing of a general linear hypothesis, the power function depends on the distribution of a quantity

$$F' = \sum_1^m (q_i - c_i)^2, \tag{22}$$

where $q_i = \frac{x_i}{\sqrt{r}}$, x_i being independently normally distributed with mean 0 and variance 1, and r having the χ_n^2 distribution, independently of the x_i. The c_i are real constants.

Let

$$\zeta = \sum_1^m c_i x_i \Big/ \sqrt{\sum_1^m c_i^2} \tag{23}$$

$$\begin{aligned} \chi^2 &= \sum_1^m (x_i - c_i\zeta)^2 = \sum_1^m x_i^2 - \zeta^2 \\ &= \sum_1^m (x_i - c_i\sqrt{r})^2 - \left(\zeta - \sqrt{r}\sqrt{\sum_1^m c_i^2}\right)^2. \end{aligned} \tag{24}$$

Now, $\sum_1^m (x_i - c_i\zeta)^2$ is a quadratic form of rank $m - 1$ since the $x_i - c_i\zeta$ are subject to one linear homogeneous restriction, namely $\sum_i c_i(x_i - c_i\zeta) = 0$. Also ζ^2 is of rank 1, and $\chi^2 + \zeta^2 = \sum_1^m x_i^2$ so that, by Cochran's Theorem, χ^2 and ζ^2 are independently distributed as χ_{m-1}^2 and χ_1^2 respectively. Thus there exist $y_1 \cdots y_m$, independently normally distributed with mean 0 and variance 1 such that

$$\begin{aligned} \chi^2 &= y_2^2 + \cdots + y_m^2 \\ \zeta^2 &= y_1^2. \end{aligned} \tag{25}$$

Let $u_i = \frac{y_i}{\sqrt{r}}$. Then the joint distribution of $u_1 \cdots u_m$ is given by

$$\begin{aligned} P\{u_1 < \tau_1, \cdots, u_m < \tau_m\} &= \frac{1}{(\sqrt{2\pi})^m} \frac{1}{(\sqrt{2})^n \Gamma(\frac{1}{2}n)} \\ &\times \int_0^\infty e^{-\frac{1}{2}r} r^{\frac{1}{2}(n-2)}\, dr \int_{-\infty}^{\tau_1\sqrt{r}} \cdots \int_{-\infty}^{\tau_m\sqrt{r}} e^{-\frac{1}{2}\sum_1^m y_i^2}\, dy_1 \cdots dy_m. \end{aligned} \tag{26}$$

The density function is given by

$$\frac{\partial^m P\{u_1 < \tau_1, \cdots, u_m < \tau_m\}}{\partial \tau_1 \cdots \partial \tau_m}$$

$$\begin{aligned}
&= \frac{1}{(\sqrt{2\pi})^m} \frac{1}{(\sqrt{2})^n \Gamma(\frac{1}{2}n)} \int_0^\infty e^{-\frac{1}{2}r} r^{\frac{1}{2}(n-2)} r^{\frac{1}{2}m} e^{-\frac{1}{2}r \sum_1^m \tau_i^2} \, dr \\
&= \frac{1}{(\sqrt{2\pi})^m} \frac{1}{(\sqrt{2})^n \Gamma(\frac{1}{2}n)} \int_0^\infty e^{-\frac{1}{2}r\left(1+\sum_1^m \tau_i^2\right)} r^{\frac{1}{2}(n+m-2)} \, dr \\
&= \frac{\left(1 + \sum_1^n \tau_i^2\right)^{-\frac{1}{2}(m+n)}}{(\sqrt{\pi})^m 2^{\frac{1}{2}(m+n)} \Gamma(\frac{1}{2}n)} \int_0^\infty e^{-\frac{1}{2}\zeta} \zeta^{\frac{1}{2}(n+m-2)} \, d\zeta \\
&= \frac{\Gamma(\frac{1}{2}(n+m))}{(\sqrt{\pi})^m \Gamma(\frac{1}{2}n)} \left(1 + \sum_1^m \tau_i^2\right)^{-\frac{1}{2}(m+n)}.
\end{aligned} \tag{27}$$

Then let

$$\eta' = \frac{\zeta}{\sqrt{r}} = \frac{y_1}{\sqrt{r}} = u_1, \qquad \tau'^2 = \frac{\chi^2}{r} = u_2^2 + \cdots + u_m^2. \tag{28}$$

The joint distribution of η' and τ'^2 is thus, by (27),

$$\begin{aligned}
&P\{\eta' < \eta, \tau'^2 < \tau^2\} \\
&= \frac{\Gamma(\frac{1}{2}(m+n))}{(\sqrt{\pi})^m \Gamma(\frac{1}{2}n)} \underset{u_1<\eta,\ \sum_2^m u_i^2 < \tau^2}{\int\int \cdots \int} \left(1 + \sum_1^m u_i^2\right)^{-\frac{1}{2}(m+n)} du_1 \cdots du_m \\
&= \frac{\Gamma(\frac{1}{2}(m+n))}{(\sqrt{\pi})^m \Gamma(\frac{1}{2}n)} \underset{u_1<\eta,\ \sum_2^m y_i^2 < \tau^2/(1+u_1^2)}{\int\int \cdots \int} (1+u_1^2)^{-\frac{1}{2}(m+n)+\frac{1}{2}(m-1)} \\
&\qquad \cdot \left(1 + \sum_2^m y_i^2\right)^{-\frac{1}{2}(m+n)} du_1 \, dy_2 \cdots dy_m \\
&= \frac{\Gamma(\frac{1}{2}(m+n))}{(\sqrt{\pi})^m \Gamma(\frac{1}{2}n)} \underset{u_1<\eta,\ \sum y_i^2 < \tau^2/(1+u_1^2)}{\int\int \cdots \int} (1+u_1^2)^{-\frac{1}{2}(n+1)} \\
&\qquad \cdot \left(1 + \sum_2^m y_i^2\right)^{-\frac{1}{2}(m+n)} du_1 \, dy_2 \cdots dy_m .
\end{aligned} \tag{29}$$

In order to evaluate this integral, we use the fact that the distribution of a ratio of χ^2_{m-1} to χ^2_{n+1}, the two being independent, can be expressed in two forms, by (27) and Wilks [2], p. 114,

$$\begin{aligned}
P\{\chi^2_{m-1}/\chi^2_{n+1} < \psi\} &= \frac{\Gamma(\frac{1}{2}(m+n))}{\Gamma(\frac{1}{2}(m-1))\Gamma(\frac{1}{2}(n+1))} \int_0^\psi \varphi^{\frac{1}{2}(m-1)-1}(1+\varphi)^{-\frac{1}{2}(m+n)} \, d\varphi \\
&= \frac{\Gamma(\frac{1}{2}(m+n))}{(\sqrt{\pi})^{m-1}\Gamma(\frac{1}{2}(n+1))} \underset{\sum_1^{m-1} q_i^2 < \psi}{\int \cdots \int} \left(1 + \sum_1^{m-1} q_i^2\right)^{-\frac{1}{2}(m+n)} dq_1 \cdots dq_m ,
\end{aligned} \tag{30}$$

so that

$$P\{\eta' < \eta, \tau'^2 < \tau^2\}$$
$$= \frac{\Gamma(\frac{1}{2}(m+n))}{\sqrt{\pi}\,\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}(m-1))} \times \int_{u_1<\eta}\int_{\varphi=0}^{\varphi=\tau^2/(1+u_1^2)} (1+u_1^2)^{-\frac{1}{2}(n+1)}\varphi^{\frac{1}{2}(m-3)}(1+\varphi)^{-\frac{1}{2}(m+n)}\,d\varphi\,du_1$$

$$(31)\quad = \frac{\Gamma(\frac{1}{2}(m+n))}{\sqrt{\pi}\,\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}(m-1))} \times \int_{u=-\infty}^{\eta}\int_{\zeta=0}^{\tau^2}(1+u^2)^{-\frac{1}{2}(n+1)}\zeta^{\frac{1}{2}(m-3)}\left(1+\frac{\zeta}{1+u^2}\right)(1+u^2)^{-\frac{1}{2}(m-3)-1}\,d\zeta\,du$$
$$= \frac{\Gamma(\frac{1}{2}(m+n))}{\sqrt{\pi}\,\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}(m-1))}\int_{u_1=-\infty}^{\eta}\int_{\zeta=0}^{\tau^2}\zeta^{\frac{1}{2}(m-3)}(1+u^2+\zeta)^{-\frac{1}{2}(m+n)}\,d\zeta\,du.$$

Now we wish to find the distribution of

$$F' = \sum_1^m (t_i - c_i)^2$$
$$(32)\quad = \frac{\sum_1^m (x_i - c_i\sqrt{r})^2}{r} = \frac{\chi^2}{r} + \frac{(\zeta - \sqrt{r}\sqrt{\Sigma c_i^2})^2}{r}$$
$$= \tau'^2 + (\eta' - \sqrt{\Sigma c_i^2})^2.$$

Carrying out the transformation (32), it is found that the joint density function of η' and F' is

$$p(\eta', F')\,d\eta'\,dF'$$
$$= \frac{\Gamma(\frac{1}{2}(m+n))}{\sqrt{\pi}\,\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}(m-1))}[F' - (\eta' - \sqrt{\Sigma c_i^2})^2]^{\frac{1}{2}(m-3)}$$
$$(33)\quad \times [1 + \eta'^2 + F' - (\eta' - \sqrt{\Sigma c_i^2})^2]^{-\frac{1}{2}(m+n)}\,d\eta'\,dF'$$
$$= \frac{\Gamma(\frac{1}{2}(m+n))}{\sqrt{\pi}\,\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}(m-1))}[F' - \rho^2]^{\frac{1}{2}(m-3)} \times [1 + F' + 2\rho\sqrt{\Sigma c_i^2} + \Sigma c_i^2]^{-\frac{1}{2}(m+n)}\,d\rho\,dF',$$

where $\rho = \eta' - \sqrt{\Sigma c_i^2}$. In order to obtain the distribution of F' we must integrate out ρ over $-\sqrt{F} < \rho < \sqrt{F}$, obtaining

$$P\{F' < T\} = \Phi_{m,n}(T, \Sigma c_i^2)$$
$$(34)\quad = \frac{\Gamma(\frac{1}{2}(m+n))}{\sqrt{\pi}\,\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}(m-1))} \times \int_{F'=0}^{T}\int_{\rho=-\sqrt{F'}}^{\sqrt{F'}}[F' - \rho^2]^{\frac{1}{2}(m-3)}[1 + F' + 2\rho\sqrt{\Sigma c_i^2} + \Sigma c_i^2]^{-\frac{1}{2}(m+n)}\,d\rho\,dF'.$$

In the case $\Sigma c_i^2 = 0$, (34) reduces to the distribution of the ratio χ_m^2/χ_n^2.

4. Test of a linear hypothesis. In this case the power of the test usually employed is affected not only by the variance, but also by the values of the predictors. In order to avoid this difficulty, it will be assumed that only a predetermined number of different sets of predictors are used, and that these sets are repeated as a whole, as many times as is necessary. This covers, in particular, the replication of orthogonal designs for the analysis of variance.

Let y_{ij}, $i = 1 \cdots m$, $j = 1, 2, \cdots$ be independently normally distributed with means

$$Ey_{ij} = \sum_{k=1}^{\mu} a_k x_{ki}, \qquad \mu \leq m, \qquad \text{rank } (x_{ki}) = \mu, \tag{35}$$

and variance σ^2, the x_{ki} being given in advance, σ^2 and a_k unknown. We wish to test H_0: $\sum_{k=1}^{\mu} c_{lk} a_k = c_{l0}$, $l = 1 \cdots r \leq \mu$, where we may suppose equations (36) linearly independent, the c_{lk} being given constants. It will be convenient to reduce this to a canonical form, as in Tang [3] First, by a non-singular linear transformation

$$x_{ki} = \sum_{l=1}^{\mu} b_{kl} z_{li} \tag{37}$$

we can make

$$\sum_{i=1}^{m} \begin{pmatrix} z_{1i} \\ \cdot \\ \cdot \\ \cdot \\ z_{\mu i} \end{pmatrix} (z_{1i} \cdots z_{\mu i}) = I_\mu, \qquad \text{the } \mu \times \mu \text{ identity matrix,} \tag{38}$$

any two sets of b_{kl} that accomplish this being related by an orthogonal transformation. Then (35) becomes

$$\begin{aligned} Ey_{ij} &= \sum_{k=1}^{\mu} a_k \sum_{l=1}^{\mu} b_{kl} z_{li} \\ &= \sum_{l=1}^{\mu} \left(\sum_{k=1}^{\mu} a_k b_{kl} \right) z_{li} = \sum_{k=1}^{\mu} a_k' z_{ki}, \end{aligned} \tag{39}$$

and (34) becomes

$$\begin{aligned} c_{l0} &= \sum_{k=1}^{\mu} c_{lk} a_k = \sum_{k=1}^{\mu} c_{lk} \sum_{m=1}^{\mu} a_m' b^{mk} \\ &= \sum_{m=1}^{\mu} a_m' \sum_{k=1}^{\mu} c_{lk} b^{mk} \\ &= \sum_{m=1}^{\mu} c_{lm}' a_m', \qquad l = 1 \cdots r \leq \mu, \end{aligned} \tag{40}$$

where b^{ml} are such that $\Sigma b^{mk} b_{kl} = \delta_{ml}$, the Kronecker delta, or, in matrix notation $(b_{km})^{-1} = (b^{km})$. Next, the equations (40) can be made into an orthonormal set

$$c_{l0}'' = \sum_{m=1}^{\mu} c_{lm} a_m' \tag{41}$$

i.e., one in which

$$\sum_{m=1}^{\mu} c''_{km} c''_{lm} = \delta_{kl} \tag{42}$$

by a non-singular linear transformation on the c'_{im}. Clearly $\Sigma c''^2_{i0}$ is an invariant of (41), i.e., it does not depend upon the choice of a particular transformation (37), or of a particular transformation of the c'_{im} into c''_{im}, since, in both cases, all admissible transformations are connected by an orthogonal transformation. Then we define

$$y'_{ij} = \sum_{q=1}^{m} z_{iq} y_{qj}, \qquad i = 1, \cdots, \mu \tag{43}$$

$$y'_{ij} = \sum_{q=1}^{m} d_{iq} y_{qj}, \qquad i = \mu + 1, \cdots, m \tag{44}$$

in such a way that $\begin{pmatrix} z_{iq} \\ d_{iq} \end{pmatrix}$ is an orthogonal matrix which is possible, by **(38)**. Then

$$\begin{aligned} Ey'_{ij} &= \sum_{q=1}^{m} z_{iq} Ey_{qj} = \sum_{q=1}^{m} z_{iq} \sum_{k=1}^{\mu} z_{kq} a'_k \\ &= \sum_{k=1}^{\mu} a'_k \sum_{q=1}^{m} z_{iq} z_{kq} = a'_i \quad \text{for} \quad i = 1, \cdots, \mu, \end{aligned} \tag{45}$$

$$\begin{aligned} Ey'_{ij} &= \sum_{q=1}^{m} d_{iq} Ey_{qj} = \sum_{q=1}^{m} d_{iq} \sum_{k=1}^{\mu} z_{kq} a'_k \\ &= \sum_{k=1}^{\mu} a'_k \sum_{q=1}^{m} d_{iq} z_{kq} = 0 \quad \text{for} \quad i = \mu + 1, \cdots, m. \end{aligned} \tag{46}$$

Finally we define

$$y''_{ij} = y'_{ij}, \qquad i = \mu + 1 \cdots, m \tag{47}$$

$$y''_{ij} = \sum_{m=1}^{\mu} c_{im} y'_{mj}, \qquad i = 1, \cdot \cdot, r \tag{48}$$

$$y''_{ij} = \sum_{m=1}^{\mu} e_{im} y'_{mj}, \qquad i = r + 1, \cdot\cdot, \mu, \tag{49}$$

where the e_{im} are such that $\begin{pmatrix} c_{im} \\ e_{im} \end{pmatrix}$ is an orthogonal matrix Since the transformation applied to the y_{ij} to obtain y''_{ij} is orthogonal, the y''_{ij} are independently normally distributed with variance σ^2. Also

$$Ey''_{ij} = 0, \; i = \mu + 1, \cdot \quad, t \tag{50}$$

$$Ey''_{ij} = \sum_{m=1}^{\mu} c_{im} a'_m = c_{i0}, \qquad i = 1, \cdots, r \tag{51}$$

$$Ey''_{ij} = \sum_{m=1}^{\mu} e_{im} a'_m, \qquad i = r + 1, \cdot \cdot, \mu. \tag{52}$$

Since (50), (51), (52) were obtained from the original formulation by a non-singular linear transformation, the derivation can be reversed, which implies the equivalence of (50), (51), (52) to the problem as originally formulated.

Thus we can restate the problem in the following manner. Let y_{ij}, $i = 1, \cdots, t$, $j = 1, 2, \cdots$ be independently normally distributed with variance σ^2 and means

$$(53)\qquad \begin{aligned} Ey_{ij} &= \xi_i, \ i = 1, \cdots, \mu \\ Ey_{ij} &= 0, \ \ i = \mu + 1, \cdots, t, \ \xi_i \text{ and } \sigma^2 \text{ unknown.} \end{aligned}$$

We wish to test

$$(54)\qquad H_0: \xi_i = 0, \ i = 1, \cdots, p \leq \mu$$

the ξ_i for $i = p + 1 \cdots \mu$ and σ^2 being nuisance parameters.

Obtain a first sample y_{ij}, $i = 1, \cdots, t$, $j = 1, \cdots, n_0$. Estimate the variance by

$$(55)\qquad s^2 = \frac{1}{n_0 t - \mu}\left\{\sum_{j=1}^{n_0}\sum_{i=1}^{t} y_{ij}^2 - \frac{1}{n_0}\sum_{i=1}^{\mu}\left(\sum_{j=1}^{n_0} y_{ij}\right)^2\right\}.$$

Let z be a predetermined constant, and n be defined by

$$(56)\qquad n = \max\left\{\left[\frac{s^2}{z}\right] + 1, n_0 + 1\right\}.$$

After s^2 has been obtained, determine a set of real numbers, $a_1 \cdots a_n$, in accordance with a preassigned rule, so as to satisfy

$$(57)\qquad \begin{aligned} \Sigma a_j &= 1 \\ s^2 \Sigma a_j^2 &= z \\ a_1 = \cdots &= a_{n_0}. \end{aligned}$$

Then

$$(58)\qquad F' = \frac{\sum_{i=1}^{p}\left(\sum_{j=1}^{n} a_j y_{ij}\right)^2}{z(n_0 t - \mu)}$$

has the non-central F-distribution given by (34) with $n = n_0 t - \mu$, $m = p$ and

$$(59)\qquad \sum_1^p c_i^2 = \sum_1^p \xi_i^2/(n_0 t - \mu)z,$$

where ξ_i are the true means, allowing for the possibility that H_0 is not true. For, $(n_0 t - \mu)s^2/\sigma^2$ has the distribution of $\chi^2_{n_0 t - \mu}$, and, after it has been determined, $\sum_{j=1}^{n} a_j y_{ij} - \xi_i$, $i = 1 \cdots r$, are independently normally distributed with mean 0 and variance $\sigma^2 \Sigma a_i^2 = \sigma^2 z/s^2$, so that, given s^2, $\left(\sum_{j=1}^{n} a_j y_{ij} - \xi_i\right)/\sqrt{z}$, $i = 1 \cdots p$

are independently normally distributed with mean 0 and variance σ^2/s^2. But the random variables t_i, in section 3 are of the form $x_i/\sqrt{r}$ where the x_i are independently normally distributed with mean 0 and variance σ^2, while r/σ^2 has the $\chi^2_{n_0t-\mu}$ distribution independent of the x_i. Thus t_i can be considered to have been obtained by first selecting a stochastic variable r such that r/σ^2 has the distribution of $\chi^2_{n_0t-\mu}$ and then selecting t_i to be independently normally distributed, given r, with mean 0 and variance σ^2/r. Since r corresponds with $(n_0t - \mu)s^2$, comparing this with the above, we find that

$$\frac{\sum_{j=1}^{n} a_j y_{ij} - \xi_i}{\sqrt{z}\sqrt{n_0 t - \mu}}, \qquad i = 1 \cdots p \tag{60}$$

have the same joint distribution as the t_i. The $\dfrac{\xi_i}{\sqrt{(n_0t - s)z}}$ are constants, so that

$$F' = \frac{\sum_{i=1}^{p}\left(\sum_{j=1}^{n} a_j y_{ij}\right)^2}{z(n_0t - \mu)} = \sum_{i=1}^{p}\left\{\frac{\sum_{j=1}^{n} a_j y_{ij} - \xi_i}{\sqrt{z(n_0t - \mu)}} + \frac{\xi_i}{\sqrt{z(n_0t - \mu)}}\right\}^2 \tag{61}$$

has the same distribution (34) as $\sum_{i=1}^{p}(t_i - c_i)^2$ with $c_i = \xi_i/\sqrt{(n_0t - \mu)z}$.

The tests of significance and confidence regions are obtained by a procedure completely analogous to that used in the case of Student's hypothesis. If we define $k = F_{p,n_0t-\mu,\alpha}$ by

$$P\{F_{p,n_0t-\mu} > k\} = \alpha, \tag{62}$$

then a critical region of size α for testing H_0 is given by

$$\frac{n_0t - \mu}{p} F' > k. \tag{63}$$

Its power function is

$$1 - \beta(\xi) = 1 - \Phi_{p,n_0t-\mu}\left(k, \frac{\sum_1^p \xi_i^2}{z(n_0t - \mu)}\right). \tag{64}$$

Similarly, a confidence region for ξ_i, $i = 1 \cdots p$, of confidence coefficient $1 - \alpha$ is given by the set of all ξ_i such that

$$\frac{n_0t - \mu}{p} F'(\xi_1 \cdots \xi_p) < k, \tag{65}$$

where

$$F'(\xi_1 \cdots \xi_p) = \frac{\sum_{i=1}^{p}\left(\sum_{j=1}^{n} a_j y_{ij} - \xi_i\right)^2}{z(n_0t - \mu)}. \tag{66}$$

It is evident that this defines the interior of the hypersphere

$$\sum_{i=1}^{p}\left(\xi_i - \sum_{j=1}^{n} a_j y_{ij}\right)^2 < kzp \tag{67}$$

whose volume is independent of the variance σ^2.

The distribution of n, the required number of sets of observations for the above tests and confidence intervals is given by

$$\begin{aligned} P\{n = n_0 + 1\} &= P\left\{\frac{s^2}{z} \le n_0 + 1\right\} \\ &= P\{(n_0 t - \mu)s^2/\sigma^2 < (n_0 + 1)(n_0 t - \mu)z/\sigma^2\} \\ &= P\{\chi_\delta^2 < y\} = \frac{1}{(\sqrt{2})^\delta \Gamma(\frac{1}{2}\delta)} \int_0^y e^{-\frac{1}{2}u} u^{\frac{1}{2}\delta - 1}\, du, \end{aligned} \tag{68}$$

where

$$\begin{aligned} y &= (n_0 + 1)(n_0 t - \mu)z/\sigma^2 \\ \delta &= n_0 t - \mu \end{aligned} \tag{69}$$

and

$$\begin{aligned} P\{n = \nu\} &= P\left\{\nu < \frac{s^2}{z} + 1 < \nu + 1\right\} \\ &= P\{(\nu - 1)\delta z/\sigma^2 < \chi_\delta^2 < \nu\delta z/\sigma^2\} \\ &= \frac{1}{(\sqrt{2})^\delta \Gamma(\frac{1}{2}\delta)} \int_{(\nu-1)\delta z/\sigma^2}^{\nu\delta z/\sigma^2} e^{-\frac{1}{2}u} u^{\frac{1}{2}\delta - 1}\, du, \end{aligned} \tag{70}$$

for integral $\nu > n_0 + 1$, all other values being impossible.

Thus $E(n)$ satisfies the inequalities

$$\begin{aligned} &\frac{1}{(\sqrt{2})^\delta \Gamma(\frac{1}{2}\delta)} \left\{\int_0^y (n_0 + 1)e^{-\frac{1}{2}u} u^{\frac{1}{2}\delta - 1}\, du + \int_y^\infty e^{-\frac{1}{2}u} u^{\frac{1}{2}\delta - 1} \frac{\sigma^2 u}{\delta z}\, du\right\} \\ &< E(n) \\ &< \frac{1}{(\sqrt{2})^\delta \Gamma(\frac{1}{2}\delta)} \left\{\int_0^y (n_0 + 1)e^{-\frac{1}{2}u} u^{\frac{1}{2}\delta - 1}\, du + \int_y^\infty e^{-\frac{1}{2}u} u^{\frac{1}{2}\delta - 1} \left(\frac{\sigma^2 u}{\delta z} + 1\right) du\right\}, \end{aligned} \tag{71}$$

which can be rewritten

$$\begin{aligned} &(n_0 + 1)P\{\chi_\delta^2 < y\} + \frac{\sigma^2}{z} P\{\chi_{\delta+2}^2 > y\} \\ &< E(n) \\ &< (n_0 + 1)P\{\chi_\delta^2 < y\} + \frac{\sigma^2}{z} P\{\chi_{\delta+2}^2 > y\} + P\{\chi_\delta^2 > y\}. \end{aligned} \tag{72}$$

The modifications required to avoid wasting information are exactly analogous to those made in the case of the test for Student's hypothesis.

5. Non existence of a single-sample test for a linear hypothesis whose power is independent of the variance. The canonical form (see Tang [3]) for a linear hypothesis in the single sample case can be derived immediately from (53) and (54). Let x_i, $i = 1 \cdots n$ be independently normally distributed with means

$$(73) \qquad Ex_i = \xi_i, i = 1 \cdots p$$

$$Ex_i = 0, i = p + 1 \cdots n$$

and variance σ^2 The ξ_i and σ^2 are unknown, and we wish to test $H_0 : \xi_i = 0$, $i = 1 \cdots p$.

The most powerful test for H_0 against a given alternative $\xi_i = \xi_{i0}$, $i = 1 \cdots p$, if the variance σ^2 is known, is that based upon the probability ratio (see Neyman and Pearson [4])

$$(74) \qquad \frac{p_1}{p_0} = \frac{\dfrac{1}{(\sqrt{2\pi}\,\sigma)^n} e^{-\frac{1}{2\sigma^2}\left\{\sum_1^p (x_i - \xi_{i0})^2 + \sum_{p+1}^n x_i^2\right\}}}{\dfrac{1}{(\sqrt{2\pi}\,\sigma)^n} e^{-\frac{1}{2\sigma^2}\sum_1^n x_i^2}} = e^{-\frac{1}{2\sigma^2}\left\{\sum_1^p \xi_{i0}^2 - 2\sum_1^p \xi_{i0} x_i\right\}}.$$

Since any strictly increasing function of p_1/p_0 is equivalent for this purpose, we can use

$$(75) \qquad \varphi(x_1 \cdots x_p) = \sum_1^p \xi_{i0} x_i .$$

The critical region of size α based upon φ is given by

$$(76) \qquad W_0(\sigma) = E\left\{x \,\middle|\, \frac{\sum_1^p \xi_{i0} x_i}{\sigma \sqrt{\sum_1^p \xi_{i0}^2}} > s\right\},$$

where

$$(77) \qquad \frac{1}{\sqrt{2\pi}} \int_s^\infty e^{-\frac{1}{2}x^2}\, dx = \alpha,$$

since, under H_0, $\sum_1^p \xi_{i0} x_i$ is normally distributed with mean 0 and variance $\sigma^2 \sum_1^p \xi_{i0}^2$. Under H_1, $\sum_1^p \xi_{i0} x_i$ is normally distributed with mean $\sum_1^p \xi_{i0}^2$ and

variance $\sigma^2 \sum_1^p \xi_{i0}^2$. Thus the power of the test for the alternative H_1 as a function of σ^2 is

$$
\begin{aligned}
1 - \beta_0(\sigma) &= P\{x \in W_0(\sigma) \mid \xi_i = \xi_{i0}, \sigma^2\} \\
&= P\left\{ \frac{\sum_1^p \xi_{i0} x_i - \sum_1^p \xi_{i0}^2}{\sigma \sqrt{\sum_1^p \xi_{i0}^2}} > z - \frac{\sqrt{\sum_1^p \xi_{i0}^2}}{\sigma} \right\} \\
&= \frac{1}{\sqrt{2\pi}} \int_{z - \frac{\sqrt{\Sigma \xi_{i0}^2}}{\sigma}}^{\infty} e^{-\frac{1}{2}x^2}\, dx.
\end{aligned}
\tag{78}
$$

Now let us suppose there exists a test based on the critical region W of size α whose power $1 - \beta$ is independent of σ^2. Since $W_0(\sigma)$ is the best critical region of size α for any σ we must have

$$
1 - \beta \leq 1 - \beta_0(\sigma) = \frac{1}{\sqrt{2\pi}} \int_{z - \frac{\sqrt{\Sigma \xi_{i0}^2}}{\sigma}}^{\infty} e^{-\frac{1}{2}x^2}\, dx,
\tag{79}
$$

so that

$$
1 - \beta \leq \operatorname*{g.l.b.}_{\sigma} [1 - \beta_0(\sigma)] = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-\frac{1}{2}x^2}\, dx = \alpha.
\tag{80}
$$

By interchanging H_0 and H_1 we can reverse the inequality (80), proving

$$
1 - \beta = \alpha.
\tag{81}
$$

Thus any single-sample test for a linear hypothesis whose power is independent of the variance has constant power equal to the size of the critical region.

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COMPACT COMPUTATION OF THE INVERSE OF A MATRIX

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1. Introduction. Among the most common applications of mathematics to practical problems are the solution of simultaneous equations, the evaluation of determinants, and the computation of the complete inverse, (or the complete adjugate), of a given matrix. Even with modern computing machines these are laborious, time-consuming jobs. For that reason there has been great interest in recent years in the development of so-called "compact" methods; that is, methods that eliminate all unnecessary detail, that use computing machines to do as much of the work as possible, and that only require copying the results needed in further analysis.

In 1935 a paper by one of the authors [1] and since then papers by the other author [2], [3], [4], [5], [6] and [7] have outlined a variety of compact methods and have applied them to actual problems. These papers, together with other recent contributions, such as those presented in [8], [9] and [10], have resulted in much improved and more compact techniques in the general field of the solution of linear simultaneous equations and allied topics, especially if the matrix is axi-symmetric. It is not generally recognized, however, that extension of these procedures (usually involving matrix factorization [7] [10]) can be used to compute the inverse (and adjugate) directly from the matrix factors without the necessity of the reduction of the unit matrix [11, 150] [2; 121] when the matrix is non-symmetric.

The present paper extends the use of compact methods in three ways.

(a) It presents a method of computing the inverse (and adjugate) of a symmetric or non-symmetric matrix by compact Gaussian methods without the formal reduction of an auxiliary identity matrix.

(b) It introduces the method of multiplication and subtraction with division—a modification of the method of multiplication and subtraction—and shows that the terms recorded in the compact solution are themselves determinants which are minors of the determinant of the matrix.

(c) It uses the method of multiplication and subtraction with division as a compact means of computing the exact value of any minor of the determinant of the matrix (whether symmetric or non-symmetric). It further shows how all cofactors of order $n - 1$ (constituting the adjugate) can be computed from a compact presentation of the calculations of the determinant of the matrix.

2. Gaussian methods and notation. Probably the method most generally used to solve simultaneous equations is the division method originated by Gauss [12]. Variations of this method are known as the Doolittle Method [13], the method of pivotal condensation [14], the method of single division [2; 104–112],

and the Crout method [8]. The methods as outlined by Gauss and Doolittle are applicable only to axi-symmetric matrices (common to least squares theory) while a more general presentation, applicable to non-symmetric matrices as well, has been made by more recent authors.

The compact form of this method, extended to apply to the non-symmetric matrix, used in this paper is as follows:

Given the matrix

$$a = (a_{rk}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots a_{3n} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} \cdots a_{nn} \end{bmatrix} \tag{1}$$

we compute

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots a_{1n} \\ b_{21} & a_{22.1} & a_{23.1} & \cdots a_{2n.1} \\ b_{31} & b_{32.1} & a_{33.12} & \cdots a_{3n.12} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2.1} & b_{n3.12} & b_{nn.12\cdots n-1} \end{bmatrix} \tag{2}$$

where

$$\begin{aligned} b_{r1} &= a_{r1}/a_{11} \\ a_{2k.1} &= a_{2k} - b_{21}a_{1k} \\ b_{r2.1} &= a_{r2} - b_{r1}a_{12})/a_{22.1} \\ a_{3k.12} &= a_{3k} - b_{32}a_{1k} - b_{32.1}a_{2k.1} \\ b_{r3.12} &= (a_{r3} - b_{r1}a_{13} - b_{r2.1}a_{23.1})/a_{33.12} \end{aligned} \tag{3}$$

and in general

$$\begin{aligned} a_{rk\cdot 12\cdots j} &= a_{rk\cdot 12\cdots j-1} - \frac{a_{jk\cdot 12\cdots j-1}\, a_{rj\cdot 12\cdots j-1}}{a_{jj\cdot 12\cdots j-1}}, \\ b_{rk\cdot 12\cdots j} &= \frac{a_{rk\cdot 12\cdots j}}{a_{kk\cdot 12\cdots j}}. \end{aligned} \tag{4}$$

It should be noted that Crout's presentation [8] is similar to that used here except that Crout divides the elements of each row by the leading element while we divide the elements of columns.

The notation used above, introduced by one of the authors [2], parallels that used extensively in multiple correlation and regression theory. It differs somewhat from the notation used by Gauss. See [12; 69]

Since every b is the ratio of two a's it follows that every b can be written in terms of a's so that the formulas can be written in terms of a's alone. This is what Gauss did although he used [ ]'s instead of a's. Gauss also used letters to indicate the primary subscripts and a single secondary subscript to indicate

the number of eliminations Thus our $a_{22\cdot 1}$ was written by Gauss as $[bb, 1]$ and $a_{33\cdot 12}$ appeared as $[cc, 2]$.

It is in the interest of less extensive notation and it makes our notation somewhat closer to that introduced by Gauss if we replace

$$a_{rk\cdot 12\cdots j} \quad \text{by} \quad a_{rk\cdot(j)}$$

$$b_{rk\cdot 12\cdots j} \quad \text{by} \quad b_{rk\cdot(j)}$$

This shortened notation can always be used when the secondary subscripts include *all* the integers from 1 to j In this modified notation the formulas (4) become

$$\text{(5)} \qquad \begin{aligned} a_{rk\cdot(j)} &= a_{rk\cdot(j-1)} - \frac{a_{jk\cdot(j-1)}\, a_{rj\cdot(j-1)}}{a_{jj\cdot(j-1)}} \\ b_{rk\cdot(j)} &= \frac{a_{rk\cdot(j)}}{a_{kk\cdot(j)}}. \end{aligned}$$

3. Solution by matrix factorization. The values of matrix (2) are in general not final answers to proposed problems but they are values from which final answers can be computed. The matrix (2) exhibits essentially both the triangular matrix of the $a_{rk\cdot(j)}$ which we call $\mathfrak{t}$ and the triangular matrix $b_{rk\cdot(j)}$ which we call $\mathfrak{s}$. (The diagonal entries of the $\mathfrak{s}$ matrix are all unity and do not appear.) Hence (2) is really $\mathfrak{s} - \mathfrak{J} + \mathfrak{t}$.

A basic property, useful in most problems involving the use of (2), is that $\mathfrak{s}$ and $\mathfrak{t}$ are factors of $\mathfrak{a}$. Thus

$$\text{(6)} \qquad \mathfrak{a} = \mathfrak{s}\mathfrak{t} \quad \text{and} \quad \mathfrak{a} - \mathfrak{s}\mathfrak{t} = 0.$$

That this is true in the symmetric case was proved in an earlier paper [7, 85]. That this is also true for the non-symmetric case is now shown in a similar manner.

Let $\mathfrak{t}_1$ be a matrix (n by n) with the first row composed of elements a_{1k} and all other elements 0 Let $\mathfrak{s}_1$ be a similar matrix with first column elements $b_{r1} = \frac{a_{r1}}{a_{11}}$ and all other elements 0 Then $\mathfrak{a} - \mathfrak{s}_1\mathfrak{t}_1 = \mathfrak{a}_1 = (a_{rk\cdot 1})$ is a matrix (n by n) with all elements of the first column and first row 0.

Next let $\mathfrak{t}_2$ be a matrix (n by n) with the second row elements $a_{2k\cdot 1}$ and all other elements 0. Let $\mathfrak{s}_2$ be a matrix (n by n) with second column elements $b_{r2\cdot 1}$ and all other elements 0. Then $\mathfrak{a}_1 - \mathfrak{s}_2\mathfrak{t}_2 = \mathfrak{a}_2 = (a_{rk\cdot(2)})$ is a matrix (n by n) with each element of the first two columns and first two rows equal to 0

This process is continued through n successive steps, an additional row and column being made identically zero at each step. We have then

$$\text{(7)} \qquad \mathfrak{a} - \mathfrak{s}_1\mathfrak{t}_1 - \mathfrak{s}_2\mathfrak{t}_2 - \cdots - \mathfrak{s}_n\mathfrak{t}_n = \mathfrak{a}_{n+1} = 0$$

Now consider the triangular matrix

$$\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3 + \cdots + \mathfrak{t}_n$$

with its rows composed of the non-zero rows of $\mathfrak{t}$. Consider also the triangular matrix $\mathfrak{s} = \mathfrak{s}_1 + \mathfrak{s}_2 + \cdots + \mathfrak{s}_n$. Then $\mathfrak{s}\mathfrak{t} = \mathfrak{s}_1\mathfrak{t}_1 + \mathfrak{s}_2\mathfrak{t}_2 + \cdots + \mathfrak{s}_n\mathfrak{t}_n$ since $\mathfrak{s}_i\mathfrak{t}_j = 0$ for $i \neq j$; and (7) becomes

$$\mathfrak{a} - \mathfrak{s}\mathfrak{t} = 0 \quad \text{or} \quad \mathfrak{a} = \mathfrak{s}\mathfrak{t}.$$

4. Gaussian computation of inverse (and adjugate) without formal reduction of auxiliary identity matrix. The inverse of $\mathfrak{a}$, $\mathfrak{a}^{-1} = \mathfrak{C} = (c_{rk})$ can be calculated directly from the matrices $\mathfrak{s}$ and $\mathfrak{t}$ of (2). The adjugate $\mathfrak{D} = (d_{rk})$ can be calculated by multiplication by the determinant of the matrix and this can be calculated by the well known formula

$$\Delta = a_{11}a_{22\cdot 1}a_{33\cdot(2)} \cdots a_{nn\cdot(n-1)}. \tag{8}$$

The theory is presented in some detail and illustrated for the case $n = 4$ after which a more general matrix presentation is given. The matrix equation $\mathfrak{a}\mathfrak{C} = \mathfrak{I}$ is equivalent to the following 4^2 simultaneous equations in the 4^2 unknowns (c_{rk}):

$$\begin{array}{lcccc}
 & k=1 & k=2 & k=3 & k=4 \\
a_{11}\,c_{1k} + a_{12}\,c_{2k} + a_{13}\,c_{3k} + a_{14}\,c_{4k} = & 1 & 0 & 0 & 0 \\
a_{21}\,c_{1k} + a_{22}\,c_{2k} + a_{23}\,c_{3k} + a_{24}\,c_{4k} = & 0 & 1 & 0 & 0 \\
a_{31}\,c_{1k} + a_{32}\,c_{2k} + a_{33}\,c_{3k} + a_{34}\,c_{4k} = & 0 & 0 & 1 & 0 \\
a_{41}\,c_{1k} + a_{42}\,c_{2k} + a_{43}\,c_{3k} + a_{44}\,c_{4k} = & 0 & 0 & 0 & 1
\end{array} \tag{9}$$

Now since $\mathfrak{C}\mathfrak{a} = \mathfrak{I}$ also we have $\mathfrak{a}'\mathfrak{C}' = \mathfrak{I}$ and there results another set of 4^2 equations in the 4^2 unknowns (c_{rk}).

$$\begin{array}{lcccc}
 & r=1 & r=2 & r=3 & r=4 \\
a_{11}\,c_{r1} + a_{21}\,c_{r2} + a_{31}\,c_{r3} + a_{41}\,c_{r4} = & 1 & 0 & 0 & 0 \\
a_{12}\,c_{r1} + a_{22}\,c_{r2} + a_{32}\,c_{r3} + a_{42}\,c_{r4} = & 0 & 1 & 0 & 0 \\
a_{13}\,c_{r1} + a_{23}\,c_{r2} + a_{33}\,c_{r3} + a_{43}\,c_{r4} = & 0 & 0 & 1 & 0 \\
a_{14}\,c_{r1} + a_{24}\,c_{r2} + a_{34}\,c_{r3} + a_{44}\,c_{r4} = & 0 & 0 & 0 & 1
\end{array} \tag{10}$$

Fisher [11; 150] has shown that the equations (9) could be solved by reducing the unit matrix on the right. One of the authors has shown how to calculate the inverse of a symmetric matrix by Gaussian methods without reducing the unit matrix [1]. We now show how to reduce the non-symmetric matrix similarly. By the same process used in getting from matrix (1) to matrix (2), we can reduce the 4^2 equations of (9) to the 4^2 auxiliary equations below.

$$\begin{array}{rcccc}
 & k=1 & k=2 & k=3 & k=4 \\
a_{11}\,c_{1k} + a_{12}\,c_{2k} + a_{13}\,c_{3k} + a_{14}\,c_{4k} = & 1 & 0 & 0 & 0 \\
a_{22\cdot 1}\,c_{2k} + a_{23\cdot 1}\,c_{3k} + a_{24\cdot 1}\,c_{4k} = & * & 1 & 0 & 0 \\
a_{33\cdot(2)}\,c_{3k} + a_{34\cdot(2)}\,c_{4k} = & * & * & 1 & 0 \\
a_{44\cdot(3)}\,c_{4k} = & * & * & * & 1
\end{array} \tag{11}$$

The terms marked * can be computed by the process. However if we do not compute these terms we have ten equations with the right hand terms either 1 or 0.

In a similar way the 4^2 equations of (10) can be reduced to the 4^2 auxiliary equations below. As above we may neglect the calculation of the diagonal terms, and of all terms below the diagonal, and still have six equations (with terms on the right zero).

$$
(12)\qquad
\begin{array}{rcccc}
 & r=1 & r=2 & r=3 & r=4 \\
c_{r1} + b_{21}\, c_{r2} + b_{31}\, c_{r3} + b_{41}\, c_{r4} = & * & 0 & 0 & 0 \\
c_{r2} + b_{32\cdot 1}\, c_{r3} + b_{42\cdot 1}\, c_{r4} = & * & * & 0 & 0 \\
c_{r3} + b_{43\cdot(2)}\, c_{r4} = & * & * & * & 0 \\
c_{r4} = & * & * & * & *
\end{array}
$$

The ten equations of (11) with the six equations of (12) are sufficient for determining the inverse matrix Solve (11) for $k = 4$; then solve (12) for $r = 4$; then solve (11) for $k = 3$; then solve (12) for $r = 3$; etc. Each equation can be solved completely on the machine to give a value of a c_{rk}.

It should be noted that Gaussian methods are approximation methods since they are division methods. For a discussion and treatment of the errors resulting the reader is referred to papers by Hotelling [9] and Satterthwaite [10] to which further reference is made in the next section.

Different forms for presentation of the results may be used. We suggest the following form which presents first the matrix (1), then the terms of the matrix (2). The terms of the matrix $\mathfrak{C}'$ are then computed by (11) and (12) and placed diagonally adjacent to the terms of (2). The transpose of $\mathfrak{C}$ is used so that the check multiplication by $\mathfrak{a}$ may be most easily accomplished. The result of this multiplication which next appears shows that the computed value of $\mathfrak{a}$ is correct to three places. The final matrix of Table I gives the value of the adjugate, $\mathfrak{D}$, as found by multiplying each element of the inverse by $(26)(52\,308)(39.356)(43.071) = 2{,}305{,}300$ (to five places).

It is possible to check the accuracy of the entries of each row and column of the matrix (2) separately by using a check sum to the right of each row and at the bottom of each column. We have not taken the space to show check sums and they are not particularly needed after one gets a little practice with the method. In any case $\mathfrak{a}\mathfrak{a}^{-1}$ should be computed as a final check.

A more general matrix presentation results from the use of (6). The matrix equation $\mathfrak{a}\mathfrak{C} = \mathfrak{I}$ becomes $\mathfrak{s}\mathfrak{t}\mathfrak{C} = \mathfrak{I}$ and hence the auxiliary equation becomes

$$
(13)\qquad \mathfrak{t}\mathfrak{C} = \mathfrak{s}^{-1}.
$$

Now since $\mathfrak{s}$ is triangular with unit diagonal terms and zeros above the diagonal, it follows that $\mathfrak{s}^{-1}$ also has unit diagonal terms with zeros above the diagonal. Hence we can select $\dfrac{n(n+1)}{2}$ equations from the n^2 equation of (13) which demand no further knowledge of the entries of $\mathfrak{s}^{-1}$. A similar treatment of the matrix equation $\mathfrak{a}'\mathfrak{C}' = \mathfrak{I}$, $\mathfrak{t}'\mathfrak{s}'\mathfrak{C}' = \mathfrak{I}$ and

$$
(14)\qquad \mathfrak{s}'\mathfrak{C}' = (\mathfrak{t}')^{-1}
$$

yields $\dfrac{n(n-1)}{2}$ equations involving zero terms of $(\mathfrak{t}')^{-1}$ These two sets of

equations taken together in the proper order are sufficient for calculating the n^2 values in the inverse.

It may be of interest to note that this is also a procedure for calculating $\mathfrak{t}^{-1}\mathfrak{s}^{-1}$ when $\mathfrak{t}$ and $\mathfrak{s}$ are known without the calculation of $\mathfrak{t}^{-1}$ and $\mathfrak{s}^{-1}$ separately since

$$\mathfrak{C} = \mathfrak{a}^{-1} = \mathfrak{t}^{-1}\mathfrak{s}^{-1}. \tag{15}$$

5. The method of multiplication and subtraction with division. We now present a different method, based upon the work of Hermite [15] and Chiò [16]

TABLE I

Suggested form for calculation

26		−10		15		32	
19		45		−14		−8	
−12		16		27		13	
32		29		−35		28	
26		−10		15		32	
	.02873		−.00696		.01825		−.00283
.73077		52.308		−24.962		−31.385	
	.02436		.01239		.01440		−.02267
−.46154		21765		39 356		34 600	
	−.02302		.01572		00791		.01991
1.23077		.78970		−.85753		43.071	
	−.01519		00419		−.02041		02322
	1.000		0.000		0 000		0 000
	0.000		1.000		0.000		0.000
	0.000		0.000		1.000		0.000
	0.000		0.000		0.000		1.000
66231		−16045		42072		−6524	
56157		28563		33196		52261	
−53068		36239		18235		45899	
−35018		9659		−47051		53529	

together with important modifications suggested by the work of Dodgson [17]. Current presentations of the basic method include the "method of condensation" [18; 45–48] and in compact forms, the "method of multiplication and subtraction" of one of the authors [2; 197–202].

In Gaussian methods we *divide* each element of a column by the leading (diagonal) element of that column. In the method of multiplication and subtraction we use the leading element as a "pivot" forming a number of two-rowed determinants. Thus we use the leading elements as *multipliers* rather than as divisors. No divisions are made in this method. This is a very real advantage when the elements of the original matrix contain only two (or three)

digits each and when $n < 7$ (or 5). In such cases we can use this method to compute *exactly* the values of any minor of the determinant of the matrix and even the adjugate itself.

It is perhaps well to mention here that error control is difficult with division (Gaussian) methods. Even if many significant places are carried the errors may be significant, cumulative, and difficult to measure. The techniques suggested by the papers of Hotelling [9] and Satterthwaite [10] are most useful in developing error control in matrix calculation. However, where accuracy is important, and when the number of digits is not excessive, there appears to be merit in calculating the exact values.

In the method of multiplication and subtraction, we compute from the matrix (1) the following matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots a_{1n} \\ a_{21} & A_{22\cdot1} & A_{23\cdot1} & \cdots A_{2n\cdot1} \\ a_{31} & A_{32\cdot1} & A_{33\cdot(2)} & \cdots A_{3n\cdot(2)} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & A_{n2\cdot1} & A_{n3\cdot(2)} & \cdots A_{nn\cdot(n-1)} \end{bmatrix} \tag{16}$$

where

$$\begin{aligned} A_{rk\cdot1} &= a_{11}a_{rk} - a_{1k}a_{r1} \\ A_{rk\cdot(2)} &= A_{22\cdot1}A_{rk\cdot1} - A_{2k\cdot1}A_{r2\cdot1} \end{aligned} \tag{17}$$

and in general

$$A_{rk\cdot(j)} = A_{jj\cdot(j-1)}A_{rk\cdot(j-1)} - A_{jk\cdot(j-1)}A_{rj\cdot(j-1)}.$$

This notation is similar to that used in connection with Gaussian methods above.

In the method of multiplication and subtraction with division, we compute from the matrix (1) the following matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots a_{1n} \\ a_{21} & B_{22\cdot1} & B_{23\cdot1} & \cdots B_{2n\cdot1} \\ a_{31} & B_{32\cdot1} & B_{33\cdot(2)} & \cdots B_{3n\cdot(2)} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & B_{n2\cdot1} & B_{n3\cdot(2)} & \cdots B_{nn\cdot(n-1)} \end{bmatrix} \tag{18}$$

where

$$\begin{aligned} B_{rk\cdot1} &= a_{11}a_{rk} - a_{1k}a_{r1} \\ B_{rk\cdot(2)} &= \frac{B_{22\cdot1}B_{rk\cdot1} - B_{2k\cdot1}B_{r2\cdot1}}{a_{11}} \\ B_{rk\cdot(3)} &= \frac{B_{33\cdot(2)}B_{rk\cdot(2)} - B_{3k\cdot(2)}B_{r3\cdot(2)}}{B_{22\cdot1}} \end{aligned} \tag{19}$$

and in general

$$B_{rk\cdot(j)} = \frac{B_{jj\cdot(j-1)}B_{rk\cdot(j-1)} - B_{jk\cdot(j-1)}B_{rj\cdot(j-1)}}{B_{j-1,j-1\cdot(j-2)}} \tag{20}$$

with $B_{rk\cdot1}$ and $B_{rk\cdot(2)}$ as defined in (19).

In general the method calls for the calculation of entries according to the method of multiplication and subtraction but in addition calls for the division by the leading element of the second preceding row or column. Since this division must be exact, as is shown in the next section, we have at each stage a good numerical check on the work as well as an *exact* value of the entry. Furthermore it is shown in the next section that the value of $B_{rk\cdot(j)}$ is the exact value of the determinant

$$(21)\qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1j} & a_{1k} \\ a_{21} & a_{22} & a_{23} \cdots a_{2j} & a_{2k} \\ a_{31} & a_{32} & a_{33} \cdots a_{3j} & a_{3k} \\ \cdots & \cdots & \cdots & \cdot \\ a_{j1} & a_{j2} & a_{j3} \cdots a_{jj} & a_{jk} \\ a_{r1} & a_{r2} & a_{r3} \cdots a_{rj} & a_{rk} \end{vmatrix}$$

All the recorded entries (themselves values of determinants) are calculated on the machine. The only limitation is the number of places the machine provides. For the trivial problems (composed of small integers) found in most texts of College Algebra, one can calculate the values readily without machines. For example the determinant

$$\begin{vmatrix} 2 & 1 & -3 & 4 \\ 3 & 2 & 2 & 1 \\ -2 & -1 & 1 & 3 \\ 4 & -3 & 2 & 1 \end{vmatrix} \quad \text{yields at once} \quad \begin{vmatrix} 2 & 1 & -3 & 4 \\ 3 & 1 & 13 & -10 \\ -2 & 0 & -2 & 7 \\ 4 & -10 & 73 & -397 \end{vmatrix}$$

and the value of Δ is -397. All the other entries are also minors of Δ.

Dodson introduced a method of multiplication and subtraction with division as early as 1866 [17]. He however used a moving pivot For our purposes it seems preferable to use a fixed pivot as we suggest in this paper.

6. Proofs of theorems involving the $B_{rk\,(j)}$.

(*a*) *First theorem.* We first prove that the numerator $B_{jj\cdot(j-1)}B_{rk\cdot(j-1)} - B_{jk\cdot(j-1)}B_{rj\,(j-1)}$ in the definition of $B_{rk\cdot(j)}$ is *exactly* divisible by the denominator $B_{j-1,j-1\cdot(j-2)}$. To do this we expand the terms of this numerator of (20) with the continued use of

$$(22)\qquad B_{rk\,(j-1)} = \frac{B_{j-1,j-1\,(j-2)}\,B_{rk\cdot(j-2)} - B_{j-1,k\,(j-2)}\,B_{r,j-1\cdot(j-2)}}{B_{j-2,j-2\,(j-3)}}$$

(which is (20) with j replaced by $j-1$) and then we multiply and cancel. It is found that $B_{j-1,j-1\cdot(j-2)}$ is a factor of all non-cancellable terms so the exact divisibility is proved.

(*b*) *Second theorem* We next prove that $B_{rk\cdot(j)}$ is the value of the determinant

(21). We illustrate first for $j = 3$ and then give a more general proof. When $j = 3$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{1k} \\ a_{21} & a_{22} & a_{23} & a_{2k} \\ a_{31} & a_{32} & a_{33} & a_{3k} \\ a_{r1} & a_{r2} & a_{r3} & a_{rk} \end{vmatrix} = \frac{1}{a_{11}^{3}} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{1k} \\ 0 & B_{22\cdot1} & B_{23\cdot1} & B_{2k\cdot1} \\ 0 & B_{32\cdot1} & B_{33\cdot1} & B_{3k\cdot1} \\ 0 & B_{r2\cdot1} & B_{r3\cdot1} & B_{rk\cdot1} \end{vmatrix}$$

$$= \frac{1}{a_{11}^{2}} \begin{vmatrix} B_{22\cdot1} & B_{23\cdot1} & B_{2k\cdot1} \\ B_{32\cdot1} & B_{33\cdot1} & B_{3k\cdot1} \\ B_{r2\cdot1} & B_{r3\cdot1} & B_{rk\cdot1} \end{vmatrix} = \frac{1}{B_{22\cdot1}} \begin{vmatrix} B_{33\cdot(2)} & B_{3k\cdot(2)} \\ B_{r3\cdot(2)} & B_{rk\cdot(2)} \end{vmatrix} = B_{rk\cdot(3)}\,.$$

In the more general case we designate the determinant (21) by $| a_{rk} |$ and reduce the order by the "condensation" method just illustrated. It is understood that the values of $B_{rk\cdot\cdot}$ used in the following proof have primary subscripts larger than secondary subscripts since the rank of the resulting determinant decreases with each condensation

$$\begin{aligned} | a_{rk} | &= \frac{1}{a_{11}^{j-1}} | B_{rk\cdot1} | = \frac{1}{B_{22\cdot1}^{j-2}} | B_{rk\cdot(2)} | \\ &= \frac{1}{B_{33\cdot(2)}^{j-3}} | B_{rk\cdot(3)} | = \cdots = \frac{1}{B_{j-1,j-1\cdot(j-2)}} | B_{rk\cdot(j-1)} | = B_{rk\cdot(j)}\,. \end{aligned} \tag{23}$$

It is to be noted that the first theorem, since each $B_{rk\cdot(j)}$ can be interpreted as a determinant by the second theorem, is a corollary of a well known theorem [19; 33]. In a conventional determinantal notation it might appear as

$$\Delta\Delta_{jk\cdot rj} = \Delta_{rk}\Delta_{jj} - \Delta_{rj}\Delta_{jk} \tag{24}$$

where the first subscripts indicate deleted rows and the second subscripts deleted columns.

(c) *Third theorem.* We next relate the values of $B_{rk\cdot(j)}$ and the values $a_{rk\cdot(j)}$ and $b_{rk\cdot(j)}$. With the use of the second theorem (23) and (8) we have

$$\frac{B_{rk\cdot(j)}}{a_{rk\cdot(j)}} = \frac{a_{11}\,a_{22\cdot1}\,a_{33\cdot(2)} \cdots a_{jj\cdot(j-1)}\,a_{rk\cdot(j)}}{a_{rk\cdot(j)}} = B_{jj\cdot(j-1)} \tag{25}$$

and with the additional use of (4)

$$\frac{B_{rk\cdot(j)}}{b_{rk\cdot(j)}} = \frac{a_{11}\,a_{22\cdot1}\,a_{33\cdot(2)} \cdots a_{jj\cdot(j-1)}\,a_{rk\cdot(j)}}{\dfrac{a_{rk\cdot(j)}}{a_{kk\cdot(j)}}} = B_{kk\cdot(j)}\,. \tag{26}$$

These formulas may be written in the form

$$\begin{aligned} B_{rk\cdot(j)} &= B_{jj\cdot(j-1)}a_{rk\cdot(j)} \\ B_{rk\cdot(j)} &= B_{kk\cdot(j)}b_{rk\cdot(j)} \end{aligned} \tag{27}$$

and since $B_{jj \cdot (j-1)}$ and $B_{j+1,j+1 \cdot (j)}$ are diagonal terms, it follows that the matrix (18) can be obtained from the matrix (2) by multiplication by diagonal matrices

(d) *Fourth Theorem.* A fourth theorem gives explicit matrix formulation to these results and shows how the values of the matrix (18) can be used in factoring the matrix (1). Now (27) and (28) can be written in the form

$$\mathfrak{T} = \mathfrak{M}_T \mathfrak{t} \tag{29}$$

$$\mathfrak{S} = \mathfrak{M}_S \mathfrak{s} \tag{30}$$

where $\mathfrak{M}_T$ is the diagonal matrix which multiplies $\mathfrak{t}$ to get $\mathfrak{T}$ and $\mathfrak{M}_S$ is the diagonal matrix which multiplies $\mathfrak{s}$ to get $\mathfrak{S}$. The values of the $\mathfrak{T}$ matrix are the values of (18) with $r \leqq k$ while the values of the $\mathfrak{S}$ matrix are the values of (18) with $r \geqq k$. The diagonal matrix $\mathfrak{M}_T$ is composed of diagonal elements $[1, a_{11}, B_{22 \cdot 1} \cdots B_{n-1,n-1 \cdot (n-2)}]$ while the matrix $\mathfrak{M}_s$ is composed of diagonal elements $[a_{11}, B_{22 \cdot 1}, B_{33 \cdot (2)} \cdots B_{nn \cdot (n-1)}]$. The basic matrix factorization equation (6) then appears as

$$\mathfrak{a} = \mathfrak{M}_s^{-1} \mathfrak{M}_T^{-1} \mathfrak{S} \mathfrak{T}. \tag{31}$$

It is to be noted that exact values of elements of all these matrices are available if the inverse diagonal matrices are written in fractional form, subject of course to practical limitations such as number of places of computing machine, etc.

7. Computation of the adjugate matrix. We now present matrix formulas which enable one to compute the adjugate of $\mathfrak{a}$ compactly with the method of multiplication and subtraction with division. If (9) is the determinant of $\mathfrak{a}$ and $\mathfrak{D}$ is the adjugate of $\mathfrak{a}$, we have

$$\begin{aligned} \mathfrak{a}\mathfrak{D} &= |a| \mathfrak{J} \\ \mathfrak{s}\mathfrak{t}\mathfrak{D} &= |a| \mathfrak{J} \\ \mathfrak{t}\mathfrak{D} &= |a| \mathfrak{s}^{-1} \\ \mathfrak{M}_t \mathfrak{t}\mathfrak{D} &= \mathfrak{M}_t |a| \mathfrak{s}^{-1} \\ \mathfrak{T}\mathfrak{D} &= \mathfrak{M}_t |a| \mathfrak{s}^{-1} \end{aligned} \tag{32}$$

and similarly

$$\begin{aligned} \mathfrak{a}'\mathfrak{D}' &= |a| \mathfrak{J} \\ \mathfrak{t}'\mathfrak{s}'\mathfrak{D}' &= |a| \mathfrak{J} \\ \mathfrak{s}'\mathfrak{D}' &= |a| (\mathfrak{t}')^{-1} \\ \mathfrak{M}_s' \mathfrak{s}'\mathfrak{D}' &= \mathfrak{M}_s |a| (\mathfrak{t}')^{-1} \\ \mathfrak{S}'\mathfrak{D}' &= \mathfrak{M}_s |a| (\mathfrak{t}')^{-1}. \end{aligned} \tag{33}$$

The computational procedure in getting the adjugate is very similar to that used in getting the inverse in section 4. $\mathfrak{T}$ and $\mathfrak{S}$ are triangular matrices while

$\mathfrak{s}^{-1}$ and $\mathfrak{t}^{-1}$ are the matrices used before The values of $\mathfrak{M}_t[1, a_{11}, B_{22\cdot 1}, \cdots B_{n-1,n-1\,(n-2)}]$ $\mathfrak{M}_s[a_{11}, B_{22-1}, B_{33\cdot(2)}, \cdots B_{nn\,(n-1)}]$ and $|a|$ are first computed by (18) so that $\mathfrak{M}_t\,|a|$ and $\mathfrak{M}_s\,|a|$ can be calculated. Without further calculation we are able to select $\dfrac{n(n+1)}{2}$ equations from the matrix equation (32) having known coefficients on the right $\left(\dfrac{n(n-1)}{2}\text{ of which are zero}\right)$ and $\dfrac{n(n-1)}{2}$ equations from the matrix equation (33) having zero coefficients on the right. These constitute the n^2 equations necessary to determine the n^2 values of d_{rk}. These values of d_{rk} can all be calculated directly on the machine and, what is more useful in discovering calculational errors, the divisions yielding the d_{rk} must be exact.

For $n = 4$ these n^2 equations are

$$
(34)\quad
\begin{array}{rcl|llll}
 & & & k=1 & k=2 & k=3 & k=4 \\
a_{11}\,d_{1k} + a_{12}\,d_{2k} + a_{13}\,d_{3k} + a_{14}\,d_{4k} & = & & |a| & 0 & 0 & 0 \\
B_{22\cdot 1}\,d_{2k} + B_{23\cdot 1}\,d_{3k} + B_{24\cdot 1}\,d_{4k} & = & & * & a_{11}\,|a| & 0 & 0 \\
B_{33\cdot(2)}\,d_{3k} + B_{34\cdot(2)}\,d_{4k} & = & & * & * & B_{22\cdot 1}\,|a| & 0 \\
B_{44\cdot 2}\,d_{4k} & = & & * & * & * & B_{33\cdot(2)}\,|a|
\end{array}
$$

$$
(35)\quad
\begin{array}{rcl|llll}
 & & & r=1 & r=2 & r=3 & r=4 \\
a_{11}\,d_{r1} + a_{21}\,d_{r2} + a_{31}\,d_{r3} + a_{41}\,d_{r4} & = & & * & 0 & 0 & 0 \\
B_{22\cdot 1}\,d_{r2} + B_{32\cdot 1}\,d_{r3} + B_{42\cdot 1}\,d_{r4} & = & & * & * & 0 & 0 \\
B_{33\cdot(2)}\,d_{r3} + B_{43\cdot(2)}\,d_{r4} & = & & * & * & * & 0
\end{array}
$$

The process is similar to that of section 4. An illustration for the case $n = 4$ is given in Table II. The matrix of the B's is directly below the matrix a and the calculated values of the elements of $\mathfrak{D}'$ (obtained by solving (34) and (35)) are placed diagonally in the cells with the B's. The values of the transpose of $\mathfrak{D}$ are used so that the check, premultiplication by $\mathfrak{a}$, is easily carried out. The next matrix in Table II exhibits $\mathfrak{a}\mathfrak{D} = |a|\,\mathfrak{I}$. The last matrix of Table II is a five decimal place approximation to $\mathfrak{C}'$ which is obtained by dividing the entries of $\mathfrak{D}'$ by $|a|$ Since we know these are the correct five decimal place values of $\mathfrak{C}'$, we may compare the corresponding values of Table I to see how much those are in error It should be noticed that the approximation to $\mathfrak{C}'$ may be readily carried to more than five decimal places if desired.

As with the Gaussian methods, it is possible here, also, to check each row and column individually by using check sums

The work necessary for the computation of the adjugate from the matrix of the B's can be shortened somewhat by the use of the fact that the adjugate is composed of the cofactors of the a_{ik}. Now the cofactors of the four terms in the lower right hand corner are $d_{n-1,n-1} = B_{n-1,n-1\,(n-2)}$; $d_{n-1,n} = -B_{n-1,n\cdot(n-2)}$; $d_{n,n-1} = -B_{n,n-1\,(n-2)}$; and $d_{nn} = B_{nn\,(n-2)}$ and these are available from the calculation of the B's though $B_{nn\,(n-2)}$ is not recorded. (See the lower right

four entries of the B's and a's in Table II above). With these four values immediately available, the use of but $n^2 - 4$ additional equations is demanded, or this additional information can be used in checking.

TABLE II

Suggested form for computation of adjugate (with check) and then inverse

26		−10		15		32	
19		45		−14		−8	
−12		16		27		13	
32		29		−35		28	
26		−10		15		32	
	66233		−16033		42069		−6503
19		1360		−649		−816	
	56151		28558		33104		−52258
−12		296		53524		47056	
	−53068		36236		18224		45899
32		1074		−45899		2305327	
	−35013		9659		−47056		53524
	2305327		0		0		0
	0		2305327		0		0
	0		0		2305327		0
	0		0		0		2305327
	02873		− 00695		.01825		−.00282
	.02436		.01239		.01440		− 02267
	− 02302		01572		.00791		01991
	−.01519		00419		−.02041		.02322

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MULTIPLE MATCHING AND RUNS BY THE SYMBOLIC METHOD

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1. Introduction. The two subjects in the title have generally been treated by distinct methods, an excellent summary of which is given by S. S. Wilks in Chapter X of [13] For two-deck matching, an appreciable simplification over the classical work of MacMahon [7], which seems to underlie the generating function used by Wilks [12] and Battin [2], has been shown by one of us [5] to follow from symbolic methods Here we give an elaboration of these methods to multiple matching and to runs.

The basis of the symbolic method in both problems has been given in [6], but for completeness a skeleton resume is given in Section 2 below. A new point is stressed the relation of coefficients in polynomials of the symbolic method to factorial moments (cf. Fréchet [4]).

The emphasis for the most part is on showing the expedition of the symbolic method in reaching known results, but in several instances new results are obtained.

2. Symbolic expressions and moments. Let $A_1, \cdots, A_n$ be arbitrary events and let $p(A_{i_1}, \cdots, A_{i_k})$ denote the joint probability of $A_{i_1}, \cdots, A_{i_k}$; let P_r be the probability that exactly r of the events occur Then

$$P_r = \sum_{k=0}^{n} (-1)^r {}_kC_r \Sigma(-1)^k p(A_{i_1}, \cdots, A_{i_k}) \tag{1}$$

and in particular

$$P_0 = \sum_{k=0}^{n} \Sigma(-1)^k p(A_{i_1}, \cdots, A_{i_k}),$$

or symbolically

$$P_0 = [1 - p(A_1)][1 - p(A_2)] \cdots [1 - p(A_n)]. \tag{2}$$

The cases to be studied will be exclusively ones where so-called *quasi-symmetry* holds, i.e., $p(A_{i_1}, \cdots, A_{i_k})$ is either 0 or a function ϕ_k of k alone. In that event (2) can be evaluated as follows: suppress all products that vanish, and form a polynomial $f(E)$ by replacing each surviving term $p(A_i)$ by E. Then $P_0 = f(E)\phi_0$ where E is a displacement operator: $E^k\phi_0 = \phi_k$.

The same polynomial $f(E)$ can also be used to obtain P_r and the moments of the distribution. From (1) we see that $P_r = f(E)\psi_0$, where $\psi_k = (-1)^r {}_kC_r\phi_k$. Again it is well known (Fréchet [4]) that the k-th factorial moment, defined by

$$M_{(k)} = \sum_{i=0}^{n} i(i-1) \cdots (i-k+1)P_i,$$

is also given by

$$M_{(k)} = k!\Sigma p(A_{i_1}, \cdots, A_{i_k}).$$

It follows that the terms of $f(E)\phi_0$ are essentially the factorial moments. More precisely, if

$$f(E) = \sum_{k=0}^{n} S_k(-E)^k,$$

then

$$M_{(k)} = k!S_k\phi_k. \tag{3}$$

3. Card matching. To avoid complications which add nothing to the fundamental idea, the case of three decks will be considered explicitly. As remarked by Battin [2], there is no loss of generality in supposing that the three decks have the same number of cards: let them be numbered from 1 to n. Let p_{ijk} denote the probability that the i-th, j-th, and k-th cards of the three decks are matched, that is, all occur in say the l-th place. The condition of quasi-symmetry is fulfilled, the (symbolic) product of k of the p's being either 0 or $\phi_k = [(n-k)!/n!]^2$.

The simplest problem is to find the probability that there be no triple matches of the form (i, i, i). Since no products of the expression

$$(1 - p_{111})(1 - p_{222}) \cdots (1 - p_{nnn})$$

vanish, the answer is $(1 - E)^n\phi_0$, in agreement with Anderson [1] (cf. also problem E 589 in the *American Mathematical Monthly*, p. 512, 1943; solution by John Riordan, p. 287, 1944).

Suppose now that the decks are given compositions in the usual fashion by having a_1, b_1, c_1 aces respectively, a_2, b_2, c_2 deuces, etc. We may number the cards so that $1, \cdots, a_1$ are aces, $a_1 + 1, \cdots, a_1 + a_2$ are deuces, and similarly in the other decks. The probability of precisely r matches among cards of the same denomination is then given by

$$F(a_1, b_1, c_1)F(a_2, b_2, c_2) \cdots \psi_0, \tag{4}$$

where

$$F(a, b, c) = \Pi(1 - p_{ijk})$$

the symbolic product being taken over ranges $i = 1, \cdots, a, j = 1, \cdots, b, k = 1, \cdots, c$.

A simple combinatorial argument reveals that

$$F(a, b, c) = \Sigma_t(a)_t(b)_t(c)_t(-E)^t/t! \tag{5}$$

where $(a)_t = a(a-1) \cdots (a-t+1)$ is the Jordan factorial notation. The problem of matching arbitrary decks is thus compactly solved by (4) and (5).

4. Examples. When decks of explicit structure are in question, the computation of probabilities and moments reduces to straightforward algebra, as is illustrated in the three following examples

1. Suppose each of three decks has two suits of two cards each. Then, since

$$F(2, 2, 2)^2 = (1 - 8E + 4E^2)^2 = 1 - 16E + 72E^2 - 64E^3 + 16E^4,$$

it follows that

$$\begin{aligned}(4!)^2P_0 &= (4!)^2 - 16(3!)^2 + 72(2!)^2 - 64(1!)^2 + 16(0!)^2 \\ &= 576 - 576 + 288 - 64 + 16 = 240,\end{aligned}$$

and the calculation of $(4!)^2P_r$ may be set forth as follows:

$$\begin{array}{ll} r & \\ 0 & 576 - 576 + 288 - \ \ 64 + 16 = 240 \\ 1 & \qquad\quad 576 - 576 + 192 - 64 = 128 \\ 2 & \qquad\qquad\qquad 288 - 192 + 96 = 192 \\ 3 & \qquad\qquad\qquad\qquad\quad 64 - 64 = \ \ 0 \\ 4 & \qquad\qquad\qquad\qquad\qquad\quad 16 = \ 16 \end{array}$$

each column being obtained by multiplying its first row entry by a binomial coefficient. These results may be verified readily by direct enumeration.

2. In the case of three 5 by 5 decks, the polynomial is

$$\begin{aligned}F(5, 5, 5)^5 &= (1 - 125E + 4000E^2 - 36000E^3 \\ &\qquad + 72000E^4 - 14400E^5)^5 \\ &= 1 - 625E + 176{,}250E^2 - 29{,}711{,}250E^3 \\ &\qquad + 3{,}346{,}063{,}125E^4 \cdots\end{aligned}$$

The factorial moments can be obtained using (3).

$$\begin{aligned}M_{(1)} &= 625/25^2 = 1, \\ M_{(2)} &= 2\cdot 176250/25^2\cdot 24^2 = 47/48, \\ M_{(3)} &= 7923/8464, \\ M_{(4)} &= 1784567/2048288,\end{aligned}$$

the first two in agreement with Battin [2].

3. The symbolic method can be applied to more intricate kinds of matching, as this final example shows. Suppose that the six matches represented by (123) and its permutations are forbidden, likewise the six matches represented by permutations of (456), and so on in groups of three. Then

$$(1 - p_{123})(1 - p_{132})(1 - p_{213})(1 - p_{231})(1 - p_{312})(1 - p_{321})$$
$$= 1 - 6E + 6E^2 - 2E^3,$$

and so the answer is

$$(1 - 6E + 6E^2 - 2E^3)^{n/3}.$$

The analogous problem for 4 decks has the solution

$$(1 - 24E + 108E^2 - 96E^3 + 24E^4)^{n/4}.$$

The generalization to an arbitrary number of decks involves the enumeration of Latin rectangles, in itself a formidable problem.

5. Moment formulas. It is possible to deduce from (4) and (5) fairly explicit formulas for the factorial moments. Let us define $u^{(t)} = (a)_t(b)_t(c)_t$. Then (5) may be written symbolically as

$$F(a, b, c) = \Sigma_t u^{(t)}(-E)^t/t! = \exp(-uE).$$

Writing $F(a_i, b_i, c_i) = \exp(-u_i E)$, we then have

$$P_0 = \exp[-(u_1 + u_2 + \cdots)E]\phi_0$$

$$= \Sigma_t(u_1 + u_2 + \cdots)^t \frac{(-E)^t}{t!} \phi_0,$$

or finally, if $m + 1$ decks are being matched,

$$P_0 = \Sigma_t(-)^t(u_1 + u_2 + \cdots)^t/t!(n)_t^m. \tag{6}$$

It is to be borne in mind that after expansion of $(u_1 + u_2 + \cdots)^t$ by the multinomial theorem, the term $u_1^x u_2^y u_3^z \cdots$ is replaced by $u_1^{(x)} u_2^{(y)} u_3^{(z)} \cdots$ with the u's defined as above.

By (3), factorial moments corresponding to (6) are given by

$$M_{(t)} = (u_1 + u_2 + \cdots)^t/(n)_t^m. \tag{7}$$

Thus in particular

$$n^m M_{(1)} = u_1 + u_2 + \cdots = \Sigma_i a_i b_i \cdots$$

$$n^m(n-1)^m M_{(2)} = (u_1 + u_2 + \cdots)^2$$

$$= \Sigma_i a_i(a_i - 1)b_i(b_i - 1) \cdots + 2\Sigma_{i \neq j} a_i a_j b_i b_j \cdots$$

the cases $m = 1, 2$ in agreement with Battin [2].

In the simple case where $m = 1$ (two decks), $a_i = b_i = a$ and $n = sa$, we have $u^{(t)} = (a)_t^2$ and

$$(n)_t M_{(t)} = (u + u + \cdots u)^t \tag{8}$$

with su's in the parenthesis. The right of (8) is the multi-variable polynomial of E. T. Bell [3], $Y_t(y_1, y_2, \cdots, y_t)$ with $y_k = (s)u^{(k)}$ and (s) a symbolic factorial such that $y_i y_j = (s)_2 u^{(i)} u^{(j)}$, etc. Instances of (8) may be compared with Olds [9].

Expanding (8) we obtain

$$(n)_t M_{(t)} = (s)_t [u^{(1)}]^t + {}_tC_2(s)_{t-1} u^{(2)} [u^{(1)}]^{t-2} + \cdots$$
$$= (s)_t a^{2t} + {}_tC_2(s)_{t-1} a^{2t-2}(a-1)^2 + \cdots$$

and, since $(s)_t/(n)_t \to a^{-t}$ as $n \to \infty$, it follows that $M_{(t)} \to a^t$, i.e., the limiting distribution is Poisson with mean a. As indicated in [6] one may proceed to obtain successive terms of an asymptotic series for the distribution. These results generalize to the case where $M_{(1)} = \Sigma a_i b_i/n$ approaches a finite limit as $n \to \infty$. In certain instances where $M_{(1)} \to \infty$, asymptotic normality can be proved (cf. [1] and [8]).

6. Successions and runs. As shown in [6], enumeration of permutations with a specified number of 2-successions like 12, 42, $\cdots$ may be accomplished by introduction of symbols like q_{12}, q_{42}, denoting probabilities that 1 immediately precede 2, 4 precede 2, resp. For permutations of objects a_1 of which are of one kind, a_2 of a second, $\cdots$ with $a_1 + a_2 + \cdots a_s = n$, the probability of exactly r 2-successions is ([6] p 914)

$$P_r = G(a_1)G(a_2) \cdots G(a_s)\psi_0 \tag{9}$$

with $\psi_k = (-1)^r_k C_r (n-k)!/n!$ and

$$G(a) = \sum_{t=0}^{a-1} (a)_t (a-1)_t (-E)^t/t!.$$

It is to be noted that in deriving (9), elements of the first kind are numbered 1 to a_1, of the second $a_1 + 1$ to $a_1 + a_2$, $\cdots$ and a succession occurs if either i precedes j or j precedes i with i and j in the same set.

For $s = 2$, i.e., two kinds of elements, there is a simpler formula due to Stevens [10], but for the general case (9) seems to be the only reasonably explicit solution known. In particular, for the function $F(a_1, \cdots, a_s)$ of Mood [8] which enumerates the number of permutations with no 2-successions, we have

$$F(a_1, \cdots, a_s) = n!G(a_1) \cdots G(a_s)\phi_0.$$

Factorial moments for 2-successions are given at once by (7):

$$M_{(t)} = (u_1 + u_2 + \cdots + u_s)^t/(n)_t \tag{10}$$

with $u_i^{(j)} = (a_i)_j(a_i - 1)_j$.

It is more usual to classify permutations according to the number of runs, say r', a run consisting of a succession of i like elements ($i = 1, 2, \cdots$). Since every 2-succession causes the loss of a potential run, we have $r' = n - r$, i.e. the number of runs is n diminished by the number of 2-successions. Factorial moments $\bar{M}_{(t)}$ for runs are then given by the usual formula for change of origin:

$$\bar{M}_{(t)} = \sum_{i=0}^{t} (-1)^i {}_tC_i(n-i)_{t-i} M_{(i)}. \tag{11}$$

Examples. 1. Introducing α_i for the i-th elementary symmetric function of the a's,

$$\alpha_1 = a_1 + a_2 + \cdots a_s = n,$$
$$\alpha_2 = a_1a_2 + a_1a_3 + \cdots + a_{s-1}a_s,$$
$$\alpha_3 = a_1a_2a_3 + \cdots,$$

we may derive from (10) and (11) the formula

$$\bar{M}_{(1)} = 1 + 2\alpha_2/n \tag{12}$$

for the mean number of runs. The variance σ^2, the same for runs and 2-successions, is given by

$$\sigma^2 = M_{(2)} + M_{(1)} - M_{(1)}^2 = \frac{2\alpha_2(2\alpha_2 - n) - 6n\alpha_3}{n_2(n-1)}. \tag{13}$$

For runs of two kinds of elements, formulas (12) and (13) specialize to those given by Wald and Wolfowitz [11]

2. For runs of elements of a single kind, factors in (9) pertaining to other elements are suppressed. Thus if a is written for a_1, and terms in $a_2, \cdots, a_s$ are suppressed, (9) and (10) become

$$P_r = G(a)\psi_0,$$
$$M_{(t)} = (a)_t(a-1)_t/(n)_t.$$

Moments for runs are given by

$$\bar{M}_{(t)} = \sum_{i=0}^{t} (-1)^i {}_tC_i(n-i)_{t-i}M_{(i)} = (a)_t(n-a+1)_t/(n)_t$$

in agreement with Mood [8].

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ON THE POWER FUNCTIONS OF THE E^2-TEST AND THE T^2-TEST

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1. The general linear hypothesis. Every linear hypothesis about a p-variate normal population or several such populations having common variances and covariances is reducible to the following canonical form [4]: The sample distribution, when nothing whatever has been discarded from the whole sample, being

$$(2\pi)^{-\frac{1}{2}p(m+n)} \, |\alpha_{ij}|^{\frac{1}{2}(m+n)} \exp\left\{-\tfrac{1}{2}\sum_{i,j=1}^{p}\alpha_{ij}\sum_{r=1}^{m} (y_{ir}-\eta_{ir})(y_{jr}-\eta_{jr}) - \tfrac{1}{2}\sum_{i,j=1}^{p}\alpha_{ij}\sum_{s=1}^{n} z_{is}z_{js}\right\} \Pi\, dy\, dz \tag{1}$$

$$(n \geq p),$$

where the η_{ir} and the α_{ij} are unknown, the hypothesis to be tested is

$$H:\quad \eta_{ir} = 0 \qquad (i = 1, \cdots, p;\, r = 1, \cdots, n_1,\, n_1 \leq m).$$

It is clear that the y_{ir} $(i = 1, \cdots, p;\, r = n_1+1, \cdots, m)$ can have no use. Also, the only useful quantities supplied by the set z_{is} are the statistics

$$b_{ij} = \sum_{s=1}^{n} z_{is}z_{js},$$

because the remaining quantities may be regarded as a set of angles which are independent of y_{ir} and the b_{ij} and which has a known distribution free from any unknown parameter in (1), [2]. After discarding the irrelevant y's and the angles there results the reduced sample distribution

$$K\,|\alpha_{ij}|^{\frac{1}{2}(n_1+n)}\,|b_{ij}|^{\frac{1}{2}(n-p-1)} \exp\left\{-\tfrac{1}{2}\sum_{i,j=1}^{p}\alpha_{ij} \cdot\sum_{r=1}^{n_1}(y_{ir}-\eta_{ir})(y_{jr}-\eta_{jr}) - \tfrac{1}{2}\sum_{i,j=1}^{p}\alpha_{ij}b_{ij}\right\} \Pi\, dy\, db.$$

Hereafter the indices i, j and r shall have the following ranges:

$$i, j = 1, \cdots, p, \qquad r = 1, \cdots, n_1,$$

and the convention that repetition of an index indicates summation will be adopted. Writing

$$a_{ij} = y_{ir}y_{jr}, \qquad c_{ij} = a_{ij} + b_{ij},$$

we obtain the distribution of the y_{ir} and the c_{ij}:

$$K\,|\alpha_{ij}|^{\frac{1}{2}(n_1+n)}\,|c_{ij} - a_{ij}|^{\frac{1}{2}(n-p-1)} \exp\left(-\tfrac{1}{2}\alpha_{ij}c_{ij} + \alpha_{ij}y_{ir}\eta_{jr} - \tfrac{1}{2}\alpha_{ij}\eta_{ir}\eta_{jr}\right)\Pi\, dy\, dc. \tag{2}$$

In the remaining two sections of this paper we deal exclusively with the special cases $p = 1$ and $n_1 = 1$. According as $p = 1$ or $n_1 = 1$ we shall drop the indices i and j or the index r.

The case $p = 1$. When $p = 1$, (2) reduces to

$$K\alpha^{\frac{1}{2}(n_1+n)}(c - y_r y_r)^{\frac{1}{2}n} \quad \exp\left(-\tfrac{1}{2}\alpha c + \alpha y_r \eta_r - \tfrac{1}{2}\alpha\eta_r\eta_r\right) dc\Pi\, dy.$$

Putting $y_r = c^{\frac{1}{2}} x_r$ we obtain

$$(3) \quad K\alpha^{\frac{1}{2}(n_1+n)} c^{\frac{1}{2}(n_1+n)-1}(1 - x_r x_r)^{\frac{1}{2}n-1} \exp\left(-\tfrac{1}{2}\alpha c + \alpha c^{\frac{1}{2}} x_r \eta_r - \tfrac{1}{2}\alpha\eta_r\eta_r\right) dc\Pi dx.$$

The hypothesis H is now

$$H' \colon \quad \eta_r = 0 \quad (r = 1, \cdots, n_1).$$

If w is any critical region for the rejection of H', denote by $w(c)$ the cross section of w for every fixed c. Then the power function of w is

$$(4) \quad \begin{aligned} \beta_w(\eta, \alpha) &= \beta_w(\eta_1, \cdots, \eta_{n_1}, \alpha) \\ &= K\alpha^{\frac{1}{2}(n_1+n)} e^{-\frac{1}{2}\alpha\eta_r\eta_r} \int_0^\infty c^{\frac{1}{2}(n_1+n)-1} e^{-\frac{1}{2}\alpha c}\, dc \int_{w(c)} (1 - x_r x_r)^{\frac{1}{2}n-1} e^{\alpha c^{\frac{1}{2}} x_r \eta_r}\, \Pi\, dx. \end{aligned}$$

It is known [3] that, in order to have

$$(5) \quad \beta_w(0, \alpha) = \epsilon$$

for all α, it is necessary and sufficient that

$$(6) \quad \int_{w(c)} (1 - x_r x_r)^{\frac{1}{2}n-1} \Pi\, dx = A\epsilon,$$

where A is a constant.

The E^2-test is the test based on the critical region

$$w_0 : \quad x_r x_r = c^{-\frac{1}{2}} y_r y_r = E^2 \geq \text{const.}$$

The author has proved [3] that of all the critical regions which satisfy (5) and whose power function is a function of $\alpha\eta_r\eta_r$ alone, the region w_0 is the uniformly most powerful one. This result is generalized by Wald [7], who proved that, of all the regions satisfying (5), the surface integral

$$\gamma_w(\alpha, \lambda) = \int_{\eta_r\eta_r=\lambda} \beta_w(\eta, \alpha)\, dA$$

is maximum when w is w_0. The author gives here another proof of Wald's theorem which is easier as it dispenses with the somewhat intricate Lemma 1 of Wald. From (4) we have

$$\begin{aligned} \gamma_w(\alpha, \lambda) = K\alpha^{\frac{1}{2}(n_1+n)} &\int_0^\infty c^{\frac{1}{2}(n_1+n)-1} e^{-\frac{1}{2}\alpha c}\, dc \\ &\cdot \int_{w(c)} (1 - x_r x_r)^{\frac{1}{2}n-1} \Pi\, dx \int_{\eta_r\eta_r=\lambda} \exp\left(-\tfrac{1}{2}\alpha\eta_r\eta_r + \alpha c^{\frac{1}{2}} x_r \eta_r\right) dA. \end{aligned}$$

By means of a rotation in the space of $(\eta_1, \cdots, \eta_{n_1})$ we can obtain

$$\int_{\eta_r\eta_r=\lambda} \exp\left(-\tfrac{1}{2}\alpha\eta_r\eta_r + \alpha c^{\frac{1}{2}} x_r\eta_r\right) dA$$
$$= \int_{\zeta_r\zeta_r=\lambda} \exp\left(-\tfrac{1}{2}\alpha\zeta_r\zeta_r + \alpha c^{\frac{1}{2}}(x_r x_r)^{\frac{1}{2}}\zeta_1\right) dA = \sum_{k=0}^{\infty} a_k \alpha^{2k}(c x_r x_r)^k,$$

where a_k depends only on α, k and λ. Hence

$$\gamma_w(\alpha, \lambda) = \sum_{k=0}^{\infty} b_k \int_0^{\infty} c^{\frac{1}{2}(n_1+n)-1} e^{-\frac{1}{2}\alpha c}\, dc \int_{w(c)} (x_r x_r)^k (1 - x_r x_r)^{\frac{1}{2}n-1} \Pi\, dx, \tag{7}$$

where b_k depends only on k, α and λ. Since $w(c)$ satisfies (6), it follows from a lemma of Neyman and Pearson [5] that

$$\int_{w(c)} (x_r x_r)^k (1 - x_r x_r)^{\frac{1}{2}n-1} \Pi\, dx$$

is maximum, for all c and k, when $w(c)$ is the region $x_r x_r \geq$ const., i.e. when w is itself the region $x_r x_r \geq$ const. This proves Wald's theorem.

Still another optimum property of the E^2-test may be established on using the volume integral instead of the surface integral. This is stated in the following theorem.

THEOREM 1. *Let S be any linear set and let*

$$\varphi_w(\alpha, S) = \int_{\eta_r\eta_r \in S} \beta_w(\eta, \alpha) \Pi\, d\eta.$$

Of all the regions satisfying (5), *the region w_0 has the maximum $\varphi_w(\alpha, S)$.*

For, by the same computation which leads to (7), we easily obtain

$$\varphi_w(\alpha, S) = \sum_{k=0}^{\infty} c_k \int_0^{\infty} c^{\frac{1}{2}(n_1+n)-1} e^{-\frac{1}{2}\alpha c}\, dc \int_{w(c)} (x_r x_r)^k (1 - x_r x_r)^{\frac{1}{2}n-1} \Pi\, dx,$$

where c_k depends only on k, α and S. Hence the result follows.

This theorem also contains my previous result as a consequence. For, writing

$$\beta_w(\eta, \alpha) = f(\alpha\eta_r\eta_r), \qquad \beta_{w_0}(\eta, \alpha) = f_0(\alpha\eta_r\eta_r),$$

we have

$$0 \leq \int_{\eta_r\eta_r \in S} (f_0(\alpha\eta_r\eta_r) - f(\alpha\eta_r\eta_r)) \Pi\, d\eta = \frac{\pi^{\frac{1}{2}n_1}}{\Gamma(\frac{1}{2}n_1)} \int_S t^{\frac{1}{2}n_1-1}(f_0(\alpha t) - f(\alpha t))\, dt.$$

Since S is arbitrary, we must have $f(\alpha t) \leq f_0(\alpha t)$.

The case $n_1 = 1$. When $n_1 = 1$, (2) and H become respectively

$$K \mid \alpha_{ij} \mid^{\frac{1}{2}(n+1)} \mid c_{ij} - y_i y_j \mid^{\frac{1}{2}(n-p-1)} \exp\left(-\tfrac{1}{2}\alpha_{ij}c_{ij} + \alpha_{ij}y_i\eta_j - \tfrac{1}{2}\alpha_{ij}\eta_i\eta_j\right) \Pi\, dy\, dc, \tag{8}$$

$$H'':\quad \eta_i = 0 \qquad (i = 1, \cdots, p).$$

There is a unique real matrix

$$\mathbf{T} = \begin{bmatrix} t_{11} & & & \\ t_{12} & t_{22} & & \\ \cdots & \cdots & & \\ t_{1p} & t_{2p} & \cdots & t_{pp} \end{bmatrix} \quad (t_{ii} > 0;\ \text{zeros above the principal diagonal})$$

such that $[c_{ij}] = \mathbf{T}\mathbf{T}'[2]$. Introducing the new variables $x_1, \cdots, x_p$ by means of the transformation

$$[y_1, \cdots, y_p] = [x_1, \cdots, x_p]\mathbf{T}' \tag{9}$$

with the Jacobian $|\mathbf{T}| = |c_{ij}|^{\frac{1}{2}}$ we obtain the distribution

$$\begin{aligned} f(x, c)\Pi\, dx\, dc = K\,|\alpha_{ij}|^{\frac{1}{2}(n+1)}\,|c_{ij}|^{\frac{1}{2}(n-p)}(1 - x_i x_i)^{\frac{1}{2}(n-p-1)} \\ \cdot \exp\left(-\tfrac{1}{2}\alpha_{ij}c_{ij} + \alpha_{ij}t_{ki}x_k\eta_j - \tfrac{1}{2}\alpha_{ij}\eta_i\eta_j\right)\Pi\, dx\, dc \end{aligned} \tag{10}$$

$$(k = 1, \cdots, p; \qquad t_{ki} = 0 \quad \text{when} \quad k > i).$$

If w is any region, we write

$$\beta_w(\eta, \alpha) = \beta_w(\eta_1, \cdots, \eta_p, \alpha_{11}, \alpha_{12}, \cdots, \alpha_{pp}) = \int_w f(x, c)\Pi\, dx\, dc,$$

so that $\beta_w(\eta, \alpha)$ is the power function if w serves as a critical region for rejecting H''. We have, symbolically,

$$w = D \times w(c),$$

where D is the set of points (c_{ij}) for which $[c_{ij}]$ is positive definite and $w(c)$ is the cross section of w for fixed c_{ij}. Then

$$\begin{aligned} \beta_w(\eta, \alpha) = K\,|\alpha_{ij}|^{\frac{1}{2}(n+1)}\,e^{-\frac{1}{2}\alpha_{ij}\eta_i\eta_j}\int_D |c_{ij}|^{\frac{1}{2}(n-p)}\,e^{-\frac{1}{2}\alpha_{ij}c_{ij}}\,\Pi\, dc \\ \cdot \int_{w(c)} (1 - x_i x_i)^{\frac{1}{2}(n-p-1)}\,e^{\alpha_{ij}t_{ki}x_k\eta_j}\,\Pi\, dx. \end{aligned}$$

It is known [6] that, in order to have

$$\beta_w(0, \alpha) = \epsilon \tag{11}$$

for all α_{ij}, it is necessary and sufficient that

$$\begin{aligned} \int_{w(c)} (1 - x_i x_i)^{\frac{1}{2}(n-p-1)}\,\Pi\, dx = B\epsilon, \\ \text{where } B = \int_{x_i x_i \leq 1} (1 - x_i x_i)^{\frac{1}{2}(n-p-1)}\,\Pi\, dx. \end{aligned} \tag{12}$$

The T^2-test is the test based on the critical region

$$w_0: \ x_i x_i = c^{ij}y_i y_j = T^2/(1 + T^2) \geq \text{const.}, \quad \text{or} \quad T^2 \geq \text{const.},$$

where c^{ij} is the general element of $[c_{ij}]^{-1}$ and T^2 is, except for a constant factor, Hotelling's generalization of "Student's" ratio.

In order to establish an optimum property of T^2 analogous to that of E^2 given in Theorem 1, we define, for any linear set S and any region R in the sample space,

$$\psi_R(S) = \int_{\alpha_{ij}\eta_i\eta_j \in S} \beta_R(\eta, \alpha)\Pi\, d\eta\, d\alpha.$$

$\Psi_R(S)$ does not necessarily have a finite value, and it is this fact which renders the following theorem less satisfactory than Theorem 1.

THEOREM 2. *Let ρ_p be the smallest latent root of $[c_{ij}]$ and let E be any subset of D in which ρ_p is at least equal to a fixed positive constant. Of all the critical regions w which satisfy* (11), *the region w_0 has the maximum $\psi_{wE}(S)$.*

In order to prove this theorem we need the following two lemmas.

LEMMA 1 *If c is a positive constant, the integral*

$$I = \int_{\rho_p \geq c} |c_{ij}|^{-(p+\frac{1}{2})}\,\Pi\, dc$$

has a finite value.

PROOF. Let $\rho_1, \cdots, \rho_p$ be the latent roots of $[c_{ij}]$ in the descending order of magnitude. From a known theorem [1] we get

$$I = C\int_{c\leq\rho_p\leq\cdots\leq\rho_1<\infty} (\rho_1\cdots\rho_p)^{-(p+\frac{1}{2})}\prod_{i<j}(\rho_i-\rho_j)\Pi\, d\rho$$
$$\leq C\int_c^\infty\cdots\int_c^\infty\left(\prod_{i=1}^p \rho_i^{-(i+\frac{1}{2})}\right)d\rho_1\cdots d\rho_p$$

Hence I is finite.

LEMMA 2.

$$(13)\quad \psi_{wE}(S) = \sum_{k=0}^\infty g_k\int_E |c_{ij}|^{-(p+\frac{1}{2})}\,\Pi\, dc\int_{w(c)}(1-x_ix_i)^{\frac{1}{2}(n-p-1)}(x_ix_i)^k\,\Pi\, dx$$

and $\psi_{wE}(S)$ is finite, where g_k depends only on k and S.

PROOF. Let Δ be the set of points (α_{ij}) for which $[\alpha_{ij}]$ is positive definite By (8), we have

$$\psi_{wE}(S) = K\int_{wE}|c_{ij} - y_iy_j|^{\frac{1}{2}(n-p-1)}\,\Pi\, dy\, dc\int_\Delta|\alpha_{ij}|^{\frac{1}{2}(n+1)}e^{-\frac{1}{2}c_{ij}\alpha_{ij}}J\,\Pi\, d\alpha,$$

where

$$J = \int_{\alpha_{ij}\eta_i\eta_j\in S}\exp\left(-\tfrac{1}{2}\alpha_{ij}\eta_i\eta_j + \alpha_{ij}y_i\eta_j\right)\Pi\, d\eta.$$

There is a real non-singular matrix $\mathbf{G} = [g_{ij}]$ such that $[\alpha_{ij}] = \mathbf{GG}'$. Using the transformation

$$[\eta_1, \cdots, \eta_p]\mathbf{G} = [\zeta_1, \cdots, \zeta_p],$$

whose Jacobian is $|\mathbf{G}|^{-1} = |\alpha_{ij}|^{-\frac{1}{2}}$, we have

$$J = |\alpha_{ij}|^{-\frac{1}{2}} \int_{\zeta_i\zeta_i \epsilon S} \exp\left(-\tfrac{1}{2}\zeta_i\zeta_i + g_{ji}\zeta_i y_j\right) \Pi\, d\zeta.$$

This is reducible by means of a rotation to

$$J = |\alpha_{ij}|^{-\frac{1}{2}} \int_{\tau_i\tau_i \epsilon S} \exp\left(-\tfrac{1}{2}\tau_i\tau_i + (\alpha_{ij}y_iy_j)^{\frac{1}{2}}\tau_1\right) \Pi\, d\tau \tag{14}$$
$$= |\alpha_{ij}|^{-\frac{1}{2}} \sum_{k=0}^{\infty} d_k(\alpha_{ij}y_iy_j)^k,$$

where

$$d_k = \frac{1}{(2k)!}\int_{\tau_i\tau_i\epsilon S} \tau_1^{2k} e^{-\frac{1}{2}\tau_i\tau_i} \Pi\, d\tau \le \frac{1}{(2k)!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \tau_1^{2k} e^{-\frac{1}{2}\tau_i\tau_i}\, d\tau_1 \cdots d\tau_p = \frac{(2\pi)^{\frac{1}{2}p}}{2^k k!}$$

and d_k depends only on k and S. Hence

$$\int_\Delta |\alpha_{ij}|^{\frac{1}{2}(n+1)} e^{-\frac{1}{2}c_{ij}\alpha_{ij}} J\, \Pi\, d\alpha = \sum_{k=0}^{\infty} d_k I_k,$$

where

$$I_k = \int_\Delta |\alpha_{ij}|^{\frac{1}{2}n} (\alpha_{ij}y_iy_j)^k e^{-\frac{1}{2}\alpha_{ij}c_{ij}} \Pi\, d\alpha. \tag{15}$$

Now

$$I_k = \frac{d^k}{dt^k} f(t)\Big|_{t=0},$$

where

$$f(t) = \int_\Delta |\alpha_{ij}|^{\frac{1}{2}n} e^{-\frac{1}{2}(c_{ij}-2ty_iy_j)\alpha_{ij}} \Pi\, d\alpha = K_1 |c_{ij} - 2ty_iy_j|^{-\frac{1}{2}(n+p+1)}$$
$$= K_1 |c_{ij}|^{-\frac{1}{2}(n+p+1)} (1 - 2tc^{ij}y_iy_j)^{-\frac{1}{2}(n+p+1)}$$

Hence

$$I_k = e_k |c_{ij}|^{-\frac{1}{2}(n+p+1)} (c^{ij}y_iy_j)^k, \tag{16}$$

where

$$e_k = \frac{K_1 2^k \Gamma\left(\dfrac{n+p+1}{2} + k\right)}{k!\,\Gamma\left(\dfrac{n+p+1}{2}\right)}.$$

Hence

$$\psi_{wE}(S) = K \sum_{k=0}^{\infty} d_k e_k \int_{wE} | c_{ij} |^{-\frac{1}{2}(n+p+1)} | c_{ij} - y_i y_j |^{\frac{1}{2}(n-p-1)} (c^{ij} y_i y_j)^k \Pi\, dy\, dc$$

$$= \sum_{k=0}^{\infty} g_k \int_E | c_{ij} |^{-(p+\frac{1}{2})} \Pi\, dc \int_{w(c)} (1 - x_i x_i)^{\frac{1}{2}(n-p-1)} (x_i x_i)^k \Pi\, dx,$$

where $g_k = K_1 d_k e_k$ depends only on k and S

Now

$$\int_{w(c)} (1 - x_i x_i)^{\frac{1}{2}(n-p-1)} (x_i x_i)^k \Pi\, dx \leq \int_{x_i x_i \leq 1} \Pi\, dx,$$

$$\int_E | c_{ij} |^{-(p+\frac{1}{2})} \Pi\, dc \leq \int_{\rho_p \geq c > 0} | c_{ij} |^{-\frac{1}{2}(p+\frac{1}{2})} \Pi\, dc$$

is finite by Lemma 1. Hence

$$\psi_{wE}(S) \leq \text{const.} \sum_{k=0}^{\infty} d_k e_k = \text{const.} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{n+p+1}{2} + k\right)}{(k!)^2}$$

and so $\varphi_{wE}(S)$ is finite. This proves Lemma 2.

Proof of Theorem 2. Since $\psi_{wE}(S)$ is expressible as (13) and is always finite, it follows from (12) and the Neyman-Pearson Lemma that $\psi_{wE}(S)$ is maximum when w is w_0. This proves Theorem 2.

Simaika [6] proved that of all the critical regions w which satisfy the conditions

(a) $\beta_w(0, \alpha) = \epsilon$ for all α_{ij},

(b) $\beta_w(\eta, \alpha) = f(\alpha_{ij}\eta_i\eta_j)$,

w_0 is the uniformly most powerful one. Strangely enough, this result cannot be deduced as a consequence from our Theorem 2.

The difficulty in dealing with the integral $\psi_w(S)$ is that it is not always finite. In order to have a finite integral let us consider the following:

$$\Gamma_w(\theta, S) = \int_{\alpha_{ij}\eta_i\eta_j \in S} e^{-\frac{1}{2}\theta_{ij}\alpha_{ij}} \beta_w(\eta, \alpha)\, \Pi\, d\eta\, d\alpha,$$

where $[\theta_{ij}]$ is a positive definite matrix. As an immediate consequence of Simaika's theorem we have

$$\Gamma_w(\theta, S) \leq \Gamma_{w_0}(\theta, S) \tag{17}$$

for any region w satisfying (a) and (b). Now the question arises whether (17) remains true if the condition (b) on w is removed. The following theorem answers this question in the negative.

THEOREM 3. *Let $[\theta_{ij}]$ be a positive definite matrix, $[\rho_{ij}] = [c_{ij} + \theta_{ij}]^{-1}$ and $\lambda_1, \cdots, \lambda_p$ be the roots of the equation $| c_{ij} - \lambda\theta_{ij} | = 0$ There is a function $g = g(\lambda_1, \cdots, \lambda_p)$ such that the region*

$$w_1: \ \rho_{ij} y_i y_j \geq g(\lambda_1, \cdots, \lambda_p)$$

satisfies (a) and has the maximum $\Gamma_w(\theta, S)$

PROOF. From (10) and (14) we obtain

$$\Gamma_w(\theta, S) = K \sum_{k=0}^{\infty} d_k \int_w | c_{ij} - y_i y_j |^{\frac{1}{2}(n-p-1)} \Pi \, dy \, dc$$

$$\cdot \int_\Delta | \alpha_{ij} |^{\frac{1}{2}n} (\alpha_{ij} y_i y_j)^k e^{-\frac{1}{2}(c_{ij}+\theta_{ij})\alpha_{ij}} \Pi \, d\alpha.$$

Comparing the inner integral with (15) and using (16) we get

$$\Gamma_w(\theta, S) = \sum_{k=0}^{\infty} g_k \int_w | c_{ij} + \theta_{ij} |^{-\frac{1}{2}(n+p+1)} | c_{ij} - y_i y_j |^{\frac{1}{2}(n-p-1)} (\rho_{ij} y_i y_j)^k \Pi \, dy \, dc$$

(18)
$$= \sum_{k=0}^{\infty} g_k \int_D | c_{ij} + \theta_{ij} |^{-\frac{1}{2}(n+p+1)} | c_{ij} |^{\frac{1}{2}(n-p)} \Pi \, dc$$

$$\cdot \int_{w(c)} (1 - x_i x_i)^{\frac{1}{2}(n-p-1)} (\gamma_{ij} x_i x_j)^k \Pi \, dx,$$

where $\gamma_{ij} x_i x_j$ is the result of applying the transformation (9) on $\rho_{ij} y_i y_j$. We shall show that, for every fixed set of c_{ij}, a unique number $g = g(\lambda_1, \cdots, \lambda_p)$ exists such that the region $\rho_{ij} y_i y_j = \gamma_{ij} x_i x_j \geq g$ satisfies (12), i.e.

(19)
$$\int_{\gamma_{ij} x_i x_j \geq g} (1 - x_i x_i)^{\frac{1}{2}(n-p-1)} \Pi \, dx = B\epsilon.$$

Since $[\gamma_{ij}] = T'[c_{ij} + \theta_{ij}]^{-1} T$, the latent roots of $[\gamma_{ij}]$ are $\lambda_i/(1 + \lambda_i)$ $(i = 1, \cdots, p)$. Hence by a rotation the equation (19) is reduced to

(20)
$$\int_{(\lambda_i/1+\lambda_i) \xi_i \xi_i \geq g} (1 - \xi_i \xi_i)^{\frac{1}{2}(n-p-1)} \Pi \, d\xi = B\epsilon.$$

As g increases from 0 onwards, the left member of (20) decreases steadily from B to 0. Hence there is a unique $g = g(\lambda_1, \cdots, \lambda_p)$ which satisfies (20).

For this $g(\lambda_1, \cdots, \lambda_p)$ the region w_1 satisfies (a). Hence, applying the Neyman-Pearson Lemma on (18) we obtain the result.

From Theorem 3 we learn that there actually exist other exact tests for H'' which have some optimum property not possessed by T^2, viz., the tests based on the critical regions w_1 corresponding to various values of the θ_{ij}. However, the great difficulty in numerical computation prohibits their application and the T^2-test stands out as the only test which is both simple and good.

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SOME GENERALIZATIONS OF THE THEORY OF CUMULATIVE SUMS OF RANDOM VARIABLES

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1. Introduction. In a previous paper [1] the author dealt with the following problem: Let $\{z_i\}$ ($i = 1, 2, \cdots$, ad inf.) be a sequence of independently distributed random variables each having the same distribution. Let a be a given positive constant, b a given negative constant and denote by n the smallest positive integer for which either

$$z_1 + \cdots + z_n \geq a \tag{1}$$

or

$$z_1 + \cdots + z_n \leq b \tag{2}$$

holds. The main problems treated in [1] were: (1) Derivation of the probability that the cumulative sum reaches the boundary a before the boundary b is reached; (2) Derivation of the characteristic function and the distribution function of n.

In this paper we shall consider the following more general problem: Let $K = \{k_i(z_1, \cdots, z_i)\}$ ($i = 1, 2, \cdots$, ad inf) be a given sequence of functions and let n be the smallest positive integer for which either

$$k_n(z_1, \cdots, z_n) \geq 1 \tag{3}$$

or

$$k_n(z_1, \cdots, z_n) \leq -1 \tag{4}$$

holds. No restrictions are imposed on the sequence K except that it must be such that the probability that $n < \infty$ is equal to one. The purpose of this paper is to derive some theorems concerning the probability that $k_n(z_1, \cdots, z_n) \geq 1$ and concerning the expected value of n. Obviously, the problem formulated here is a generalization of that considered in [1], since the latter can be obtained by putting $k_i(x_1, \cdots, x_i) = \frac{2}{a-b}(z_1 + \cdots + z_i) - \frac{a+b}{a-b}$.

2. The conjugate distribution of z. Let z be a random variable whose distribution is equal to the common distribution of z_i. In this section we shall introduce the notion of the conjugate distribution of z which will be used later. According to Lemma 2 in [1], under some weak restrictions on the distribution of z there exists exactly one real value $h_0 \neq 0$ such that

$$E(e^{zh_0}) = 1 \tag{5}$$

where $E(u)$ denotes the expected value of u for any random variable u.

For simplicity we shall assume that z has a continuous distribution admitting a probability density everywhere, or that z has a discrete distribution. By the probability distribution $f(z)$ of z we shall mean the probability density of z, if the distribution of z is continuous In the discrete case $f(z)$ will denote the probability that the random variable takes the value z. From (5) it follows that

$$f^*(z) = e^{zh_0}f(z) \tag{6}$$

is a probability distribution. We shall call $f^*(z)$ the conjugate distribution of z. For any random variable u we shall denote by $E^*(u)$ the expected value of u under the assumption that the distribution of z is given by $f^*(z)$. The expected values $E(u)$ and $E^*(u)$ may depend on the sequence $K = \{k_i(z_1, \cdots, z_i)\}$ ($i = 1, 2, \cdots$, ad inf.). Occasionally we shall put this dependence in evidence by writing $E(u \mid K)$ and $E^*(u \mid K)$, respectively.

3. Two theorems. In this section we shall derive two theorems. The first theorem is concerned with the probability that $k_n(z_1, \cdots, z_n) \geq 1$ and the second theorem with the expected value of n. In what follows the operator E_1 will mean conditional expected value under the restriction that $k_n(z_1, \cdots, z_n) \geq 1$ and E_2 will mean conditional expected value under the restriction that $k_n(z_1, \cdots, z_n) \leq -1$. If the distribution of z is given by $f^*(z)$, these conditional expected values will be denoted by the operators E_1^* and E_2^*, respectively.

THEOREM 1. *Let $K = \{k_i(z_1, \cdots, z_i)\}$ be a sequence such that the probability that $n < \infty$ is equal to one under both distributions $f(z)$ and $f^*(z)$. Let γ denote the probability that $k_n(z_1, \cdots, z_n) \geq 1$ when $f(z)$ is the distribution of z, and let γ^* denote the probability of the same event when $f^*(z)$ is the distribution of z. Then*

$$E_1(e^{Z_n h_0} \mid K) = \frac{\gamma^*}{\gamma}; \qquad E_2(e^{Z_n h_0} \mid K) = \frac{1 - \gamma^*}{1 - \gamma} \tag{7}$$

and

$$E_1^*(e^{-Z_n h_0} \mid K) = \frac{\gamma}{\gamma^*}; \qquad E_2^*(e^{-Z_n h_0} \mid K) = \frac{1 - \gamma}{1 - \gamma^*} \tag{8}$$

where $Z_n = z_1 + \cdots + z_n$.

PROOF: From (6) it follows that

$$e^{Z_n h_0} = \frac{f^*(z_1) \cdots f^*(z_n)}{f(z_1) \cdots f(z_n)} \tag{9}$$

and

$$e^{-Z_n h_0} = \frac{f(z_1) \cdots f(z_n)}{f^*(z_1) \cdots f^*(z_n)}. \tag{10}$$

A set $(z_1, \cdots, z_n)$ will be said to be of type 1 if and only if $-1 < k_m(z_1, \cdots, z_m) < 1$ for $m = 1, \cdots, n - 1$ and $k_n(z_1, \cdots, z_n) \geq 1$. Similarly a set $(z_1, \cdots z_n)$ will be said to be of type 2 if and only if $-1 < k_m(z_1, \cdots, z_m) < 1$ for $m = 1, \cdots, n - 1$ and $k_n(z_1, \cdots, z_n) \leq -1$.

We shall prove Theorem 1 under the assumption that the distribution of z is discrete Because of (9) we have

$$(11) \quad E_1(e^{Z_n h_0} \mid K) = E_1\left(\frac{f^*(z_1) \cdots f^*(z_n)}{f(z_1) \cdots f(z_n)} \,\middle|\, K\right) = \frac{\sum_{(z_1, \cdots, z_n)} f^*(z_1) \cdots f^*(z_n)}{\sum_{(z_1, \cdots, z_n)} f(z_1) \cdots f(z_n)}$$

where the summation is to be taken over all sets $(z_1, \cdots, z_n)$ of type 1. But the last expression is obviously equal to $\frac{\gamma^*}{\gamma}$ and, therefore, the first equation in (7) is proved. The second equation in (7) follows in the same manner if we take into account the fact that the probability that $n < \infty$ is equal to one. Similarly, equation (8) can be obtained from (10). The proof can easily be extended to the case when the distribution of z is continuous. Hence, Theorem 1 is proved

THEOREM 2. *If $Ez \neq 0$, the relation*

$$(12) \qquad E(n \mid K) = \frac{E(Z_n \mid K)}{Ez}$$

holds for any sequence $K = \{k_i(z_1, \cdots, z_i)\}$ for which one of the following two conditions is fulfilled:

(a) *There exists an integer N such that the probability that $n < N$ is equal to one.*
(b) *$E(n \mid K) < \infty$ and the first four moments of z are finite.*

PROOF: First we shall show that condition (a) implies the validity of (12) For any integer i we shall denote $z_1 + \cdots + z_i$ by Z_i. Since the probability that $n < N$ is equal to 1, we have

$$(13) \qquad E(Z_n \mid K) + E(z_{n+1} + \cdots + z_N) = EZ_N = NEz.$$

Since the conditional expected value of $(z_{n+1} + \cdots + z_N)$ for a given value of n is equal to $(N - n)Ez$, we have

$$(14) \qquad E(z_{n+1} + \cdots + z_N) = E(N - n \mid K)Ez = NEz - E(n \mid K)Ez.$$

Equation (12) follows from (13) and (14).

Now we shall show that condition (b) implies (12). Denote by P_N the probability that $n \leq N$. Let the operator E_N denote conditional expected value under the restriction that $n \leq N$, and let the operator E'_N denote conditional expected value under the restriction that $n > N$. Then we have

$$(15) \qquad P_N E_N(Z_N) + (1 - P_N)E'_N(Z_N) = E(Z_N) = NEz.$$

Since

$$\begin{aligned} &= E_N(Z_n \mid K) + E_N(z_{n+1} + \cdots + z_N \mid K) \\ (16) \quad E_N(Z_N) &= E_N(Z_n \mid K) + E_N(N - n \mid K)Ez \\ &= E_N(Z_n \mid K) + NEz - E_N(n \mid K)Ez, \end{aligned}$$

we obtain from (15)

(17) $P_N\{E_N(Z_n \mid K) + NEz - E_N(n \mid K)Ez\} + (1 - P_N)E'_N(Z_N) = NEz.$

From $E(n \mid K) < \infty$ it follows that

$$\lim_{N=\infty} (1 - P_N)N = 0. \tag{18}$$

Now we shall show that (18) implies the validity of

$$\lim_{N=\infty} (1 - P_N)E'_N(Z_N) = 0 \tag{19}$$

Let $T_N = Z_N - NEz$. Because of (18), (19) is proved if we can show that

$$\lim_{N=\infty} (1 - P_N)E'_N(T_N) = 0. \tag{20}$$

Denote by R_N the set of all points $(z_1, \cdots, z_N)$ for which $n > N$. Then the probability measure of R_N is equal to $1 - P_N$ and

$$(1 - P_N)E'_N(T_N) = \int_{R_N} T_N f(z_1) \cdots f(z_N)\, dz_1 \cdots dz_N. \tag{21}$$

Let R_N^1 be the part of R_N in which $T_N < -N$, R_N^2 the part of R_N in which $T_N > N$ and R_N^3 the part of R_N in which $-N \leq T_N \leq N$ Because of (18) we have

$$\lim_{N=\infty} \left| \int_{R_N^3} T_N f(z_1) \cdots f(z_N)\, dz_1 \cdots dz_N \right| \leq \lim_{N=\infty} (1 - P_N)N = 0. \tag{22}$$

Denote the cumulative distribution function of T_N by $F_N(T_N)$. Clearly,

$$\int_{R_N^2} T_N f(z_1) \cdots f(z_N)\, dz_1 \cdots dz_N \leq \int_N^\infty T_N\, dF_N(T_N) \leq \frac{1}{N^3} \int_N^\infty T_N^4\, dF_N(T_N). \tag{23}$$

Since the first four moments of z are finite, the 4-th moment of $\frac{T_N}{\sqrt{N}}$ converges to $3\sigma^4$ where σ is the standard deviation of z. Hence

$$\lim_{N=\infty} \int_{-\infty}^{+\infty} \frac{1}{N^2} T_N^4 dF_N(T_N) = 3\sigma^4 \tag{24}$$

From (23) and (24) it follows that

$$\lim_{N=\infty} \int_{R_N^2} T_N f(z_1) \cdots f(z_N)\, dz_1 \cdots dz_N = 0. \tag{25}$$

Similarly we can prove that

$$\lim_{N=\infty} \int_{R_N^1} T_N f(z_1) \cdots f(z_N)\, dz_1 \cdots dz_N = 0. \tag{26}$$

Equation (20) follows from (21), (22), (25) and (26). Hence (19) is proved. From (17), (18) and (19) we obtain

$$\lim_{N=\infty} P_N\{E_N(Z_n \mid K) - E_N(n \mid K)Ez\} = 0. \tag{27}$$

Since $Ez \neq 0$, $\lim P_N = 1$, $\lim E_N(n \mid K) = E(n \mid K)$ and $\lim E_N(Z_n \mid K) = E(Z_n \mid K)$, equation (12) follows from (27) Hence condition (b) implies (12) and Theorem 2 is proved.

4. Lower limit of $E(n \mid K)$. In this section we shall derive a lower limit for $E(n \mid K)$ First we shall prove the following lemma

LEMMA 1. *For any random variable u we have*

$$e^{E(u)} \leq Ee^u. \tag{28}$$

PROOF. Inequality (28) can be written as

$$1 \leq Ee^{u'} \tag{29}$$

where $u' = u - Eu$. Lemma 1 is proved if we show that (29) holds for any random variable u' whose mean is zero. Expanding $e^{u'}$ in a Taylor series around $u' = 0$, we obtain

$$e^{u'} = 1 + u' + \frac{u'^2}{2} e^{\xi(u')} \qquad \text{where } 0 \leq \xi(u') \leq u'.$$

Hence

$$Ee^{u'} = 1 + \tfrac{1}{2}Eu'^2e^{\xi(u')} \geq 1$$

and Lemma 1 is proved

Now we are able to prove the following theorem.

THEOREM 3. *Let $K = \{K_i(z_1, \cdots, z_i)\}$ be a sequence of functions such that the probability that $n < \infty$ is one under both distributions $f(z)$ and $f^*(z)$ of z. Let γ be the probability that $K_n(z_1, \cdots, z_n) \geq 1$ when $f(z)$ is the distribution of z, and let γ^* be the probability of the same event when $f^*(z)$ is the distribution of z. Then*

$$E(n \mid K) \geq \frac{1}{h_0 Ez}\left[\gamma \log \frac{\gamma^*}{\gamma} + (1 - \gamma) \log \frac{1 - \gamma^*}{1 - \gamma}\right] \tag{30}$$

and

$$E^*(n \mid K) \geq \frac{1}{h_0 Ez^*}\left[\gamma^* \log \frac{\gamma^*}{\gamma} + (1 - \gamma^*) \log \frac{1 - \gamma^*}{1 - \gamma}\right], \tag{31}$$

provided that Ez and Ez^* are not equal to zero.

PROOF: First we shall prove Theorem 3 in the case when there exists an integer N such that the probability that $n < N$ is one According to Theorem 2 we have

$$E(n \mid K) = \frac{E(Z_n \mid K)}{Ez} = \frac{1}{Ez}[\gamma E_1(Z_n \mid K) + (1 - \gamma)E_2(Z_n \mid K)]. \tag{32}$$

From Lemma 1 and Theorem 1 it follows that

$$h_0 E_1(Z_n \mid K) \leq \log \frac{\gamma^*}{\gamma} \qquad \text{and } h_0 E_2(Z_n \mid K) \leq \log \frac{1 - \gamma^*}{1 - \gamma}. \tag{33}$$

From (32) and (33) we obtain

$$
(34)\qquad \begin{aligned} h_0 Ez\, E(n \mid K) &= h_0[\gamma E_1(Z_n \mid K) \\ &\quad + (1-\gamma)E_2(Z_n \mid K)] \le \gamma \log \frac{\gamma^*}{\gamma} + (1-\gamma)\log\frac{1-\gamma^*}{1-\gamma}. \end{aligned}
$$

Inequality (30) follows from (34) if we can show that $h_0E(z) < 0$. From $Ee^{h_0 z} = 1$ and Lemma 1 it follows that $h_0E(z) \le 0$. Since $h_0 \ne 0$ and $E(z) \ne 0$, we must have $h_0E(z) < 0$. Hence (30) is proved. To prove (31) we proceed as follows: From Theorem 2 we obtain

$$
(35)\qquad -h_0Ez^*E^*(n \mid K) = -h_0[\gamma^* E_1^*(Z_n \mid K) + (1-\gamma^*)E_2^*(Z_n \mid K)].
$$

From Lemma 1 and Theorem 1 it follows that

$$
(36)\qquad \begin{aligned} -h_0[\gamma^* E_1^*(Z_n \mid K) &+ (1-\gamma^*)E_2^*(Z_n \mid K)] \\ &\le \gamma^* \log\frac{\gamma}{\gamma^*} + (1-\gamma^*)\log\frac{1-\gamma}{1-\gamma^*}. \end{aligned}
$$

From (35) and (36) we obtain

$$
(37)\qquad h_0E^*(z)E^*(n \mid K) \ge \gamma^* \log\frac{\gamma^*}{\gamma} + (1-\gamma^*)\log\frac{1-\gamma^*}{1-\gamma}.
$$

Since $E^*e^{-h_0 z} = 1$ it follows from Lemma 1 that $-h_0E^*z \le 0$. Inequality (31) follows from this and (37). Hence Theorem 3 is proved in the special case when there exists an integer N such that the probability that $n < N$ is equal to one.

To prove Theorem 3 in the general case, for any integer N let the sequence $K_N = \{k_{iN}(z_1, \cdots, z_i)\}$ be defined as follows: $k_{iN}(z_1, \cdots, z_i) = k_i(z_1, \cdots, z_i)$ for $i < N$ and $k_{iN}(z_1, \cdots, z_i) = 1$ for $i \ge N$. Denote by γ_N and γ_N^* the values of γ and γ^*, respectively, if the sequence K is replaced by K_N. Then we have

$$
(38)\qquad E(n \mid K) \ge E(n \mid K_N) \ge \frac{1}{h_0Ez}\left[\gamma_N \log\frac{\gamma_N^*}{\gamma_N} + (1-\gamma_N)\log\frac{1-\gamma_N^*}{1-\gamma_N}\right]
$$

and

$$
(39)\qquad E^*(n \mid K) \ge E^*(n \mid K_N) \ge \frac{1}{h_0Ez^*}\left[\gamma_N^* \log\frac{\gamma_N^*}{\gamma_N} + (1-\gamma_N^*)\log\frac{1-\gamma_N^*}{1-\gamma_N}\right].
$$

Since $\lim_{N=\infty} \gamma_N = \gamma$ and $\lim_{N=\infty} \gamma_N^* = \gamma^*$, inequalities (30) and (31) follow from (38) and (39). Hence the proof of Theorem 3 is completed.

5. Remarks added in proof. The results obtained in the present paper have obvious applications to sequential analysis. These applications are, however, not mentioned here, because at the time the present paper was submitted for publication, sequential analysis constituted classified material. In the meantime, the material on sequential analysis has been released and was published in

this Journal, June, 1945. The results obtained in the present paper are more general than those obtained in connection with sequential analysis. Theorem 3, in the present paper, implied the efficiency of the sequential probability ratio test discussed in Section 4 7 of the paper on sequential tests.

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ON THE DESIGN OF EXPERIMENTS FOR WEIGHING AND MAKING OTHER TYPES OF MEASUREMENTS

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1. Introduction. In a recent paper, Hotelling [1] has discussed the basic principles of the theory of the design of efficient experiments for estimating the true unknown weights of p given objects by means of a specified number N of weighings, $p \leq N$ in case the scale is free from bias and $p \leq N - 1$ if it has a bias the unknown value of which has to be estimated from the same data. He has emphasized the importance of these designs in other kinds of measurements besides weighing of objects and has called attention to the need for further mathematical research for obtaining a "comprehensive general solution." Such a solution has now been obtained in case the number of weighings N is at our choice. Some other general designs have also been given in this paper for specified values of N and p.

2. Estimation of unknown weights and efficiency of a design. Using Hotelling's notation, we may write

$$E(y_\alpha) = \sum_{i=1}^{p} x_{i\alpha} b_i \tag{1}$$

where $i = 1, 2, \cdots p$, on the assumption that there is either zero bias in the scale or the bias is known *a priori*, and $\alpha = 1, 2, \cdots N$. $E(y_\alpha)$ is the expectation of the αth weighing. For a biassed scale, we may take $i = 0, 1, 2, \cdots p$. The efficient estimate of each of the b_i's has been derived by Hotelling by the method of least squares. It is of interest to obtain these estimates by the use of the theory of linear estimation as developed by Bose [2] and Rao [3].

Assuming that $y_1, y_2, \cdots y_N$ are N stochastic variates forming a multivariate normal system with the variance and covariance matrix given by

$$u = [u_{ij}], \tag{2}$$

it follows from Rao's generalization of Markoff's theorem that the best unbiassed estimates of the b_i's are given by the solutions of the normal equations

$$X'U^{-1}XB' = X'U^{-1}Y', \tag{3}$$

where $B = [b_1b_2 \cdots b_p]$ and $Y = [y_1y_2 \cdots y_N]$, and B' and Y' denote as usual the transpose of the row vectors B and Y, i.e. column vectors.

In the present case, the assumption is that all the N stochastic variates are uncorrelated and have a common variance σ^2, so that

$$U^{-1} = \frac{1}{\sigma^2} I\,. \tag{4}$$

Hence the normal equations in (3) reduce to

$$X'XB' = X'Y', \tag{5}$$

which are exactly the same as the normal equations given by Hotelling, since

$$X'X = [a_{ij}] \tag{6}$$

where $a_{ij} = S(x_{i\alpha}x_{j\alpha})$

Let $C = [c_{ij}]$ denote the reciprocal of the matrix $X'X$, so that $V(b_i) = c_{ii}\sigma^2$ and cov $(b_ib_j) = c_{ij}\sigma^2$ Then the mean variance of the p unknowns for a design is given by

$$v_m = \frac{\sigma^2}{N} \cdot \frac{N \sum_{i=1}^{p} c_{ii}}{p}. \tag{7}$$

If the main object of the experiment is to estimate the unknowns with the least variance, the most efficient design (for a specified value of N) would be the one for which the *minimum minimorum* of σ^2/N is attained for all the p unknowns so that the mean variance in this case is σ^2/N The factor, $N \sum_{i=1}^{p} c_{ii}/p$, on the right-hand side of (7), therefore, measures the increase in variance resulting from the adoption of any design other than the most efficient design. Its reciprocal, $\dfrac{p}{N \sum_{i=1}^{p} c_{ii}}$, may appropriately be defined as the *efficiency* of a given design for providing estimates of the p unknowns This quantity will now be utilized for judging the relative precision of the general designs discussed in the subsequent paragraphs.

3. Design for $N = 2^m$, $p \leq 2^m$ (zero bias) or $p \leq 2^m - 1$ (non-zero bias). By utilizing the properties of a 2-sided m-fold completely orthogonalized Hyper-Graeco-Latin hyper-cube of the *first* order introduced by the author [4], it is easy to see that for $N = 2^m$, $p \leq 2^m$ (when there is zero bias) or $p \leq 2^m - 1$ (when there is bias), m being any positive integer, a completely orthogonalized design can be constructed with each unknown weight estimated with the minimum variance σ^2/N As remarked by Hotelling in the case of $N = 4$, $p = 4$ (for zero bias) or $p = 3$ (if there is bias), the matrix $X'X$ for this design is a scalar matrix of order $p \times p$ if there is zero bias, or of order $(p + 1) \times (p + 1)$ if there is bias, each of the diagonal elements being N. The reciprocal matrix is also a scalar matrix in which each of the diagonal elements is $1/N$ so that the estimates of all the unknowns are mutually orthogonal.

As a particular case of this general design, we may take $N = 16$, $p = 16$ (for zero bias) or $p = 15$ (if there is bias), the completely orthogonalized design for which is represented by the matrix

(8) $X =$

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\
1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \\
1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1
\end{pmatrix}$$

for which $X'X$ is a scalar matrix of order 16×16, each diagonal element being 16. Again, a completely orthogonalized design for $N = 16$, $p < 16$ (for zero bias) or $p < 15$ (if there is bias) is represented by a matrix X obtained from the matrix in (8) by omitting any $16 - p$ of its columns if there is zero bias, or $16 - p - 1$ of its columns if there is bias. In the matrix X, permutation of rows and columns is permissible and each such matrix represents a completely orthogonalized design.

For the design given by Hotelling[1] for $N = 4$, $p = 3$ (zero bias), the efficiency is 35 per cent. The completely orthogonalized design for which the efficiency is 100 per cent is represented by the matrix

$$X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \tag{9}$$

4. First design for $N = 2^m + 1$, $p \leq 2^m$ (zero bias) or $p \leq 2^m - 1$ (non-zero bias). For $N = 2^m + 1$, $p \leq 2^m$ (zero bias) or $p \leq 2^m - 1$ (if there is bias), m being any positive integer, probably the most efficient design available seems to be that represented by the matrix X obtained from the corresponding matrix

[1] The allusions here and at the end of the next section are to designs on p. 305 of the Hotelling paper [1], a passage concerned with designs subject to the restriction that the entries on the matrix be 0's and +1's only, as is necessary in many types of measurement. The more efficient designs given above, whose matrices involve −1's also, can be used only in such cases as that of weighing in a balance, where the objects under investigation can be put, some in one pan and some in the other. Such situations are considered in a different part of Hotelling's paper.

X for the general design of Section 3 above by adding a row 1, 1, $\cdots$ 1 to it. The matrix $X'X$ for this design then comes out as

$$X'X = \begin{pmatrix} N & 1 & 1 \cdots & 1 \\ 1 & N & 1 \cdots & 1 \\ 1 & 1 & N \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 \cdots & N \end{pmatrix}, \tag{10}$$

which is a symmetrical matrix of order $p \times p$ if there is zero bias, or of order $(p + 1) \times (p + 1)$ if there is bias. The variance of each unknown for this design is

$$\frac{\sigma^2}{N - \dfrac{p-1}{N+p-2}} \qquad \text{for zero bias,} \tag{11}$$

or

$$\frac{\sigma^2}{N - \dfrac{p}{N+p-1}} \qquad \text{if there is bias.} \tag{12}$$

Thus the efficiency of this design is

$$1 - \frac{p-1}{N(N+p-2)} \qquad \text{for zero bias,} \tag{13}$$

or

$$1 - \frac{p}{N(N+p-1)} \qquad \text{if there is bias.} \tag{14}$$

The loss of efficiency resulting from the adoption of this design is, therefore, $\dfrac{p-1}{N(N+p-2)}$ for zero bias or $\dfrac{p}{N(N+p-1)}$ if there is bias.

As a particular case of this, for $N = 5$, $p = 2$ (zero bias), probably the most efficient design available is specified by

$$X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}. \tag{15}$$

The variance of each unknown in this case is $\dfrac{5\sigma^2}{24}$ and the efficiency of the design is 96 per cent. For the design given by Hotelling for this case, the variance of each unknown is $\dfrac{4\sigma^2}{7}$ and the efficiency is 35 per cent. It would thus appear

that, as judged by the criterion of efficiency as defined here, the design represented by the matrix in (15) is more efficient than Hotelling's design

5. Second design for $N = 2^m + 1$, $p \leq 2^m$ (zero bias) or $p \leq 2^m - 1$ (non-zero bias). Another interesting design for these values of N and p is that represented by the matrix X obtained by adding a row 1, 0, $\cdots$ 0 to the corresponding matrix X for the general design in Section 3 above The matrix $X'X$ for this design is then the diagonal matrix

$$X'X = \begin{pmatrix} N & 0 & \cdots & 0 \\ 0 & N-1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & N-1 \end{pmatrix} \tag{16}$$

of order $p \times p$ (for zero bias) or $(p + 1) \times (p + 1)$ (for non-zero bias). As the reciprocal of this matrix is also a diagonal matrix, the estimates of all the unknowns are mutually orthogonal. The efficiency of this design is

$$\frac{(N - 1)p}{Np - 1} \qquad \text{for zero bias,} \tag{17}$$

or

$$\frac{N - 1}{N} \qquad \text{for non-zero bias} \tag{18}$$

By comparing the efficiency of the first design given in (13) and (14) with that of the second design in (17) and (18) respectively, it would appear that the efficiency of the first design is always higher than that of the second design for non-zero bias, and is also higher in the case of zero bias for $p > 1$, but equal for $p = 1$.

6. First design for $N = 2^m + r$, $p \leq 2^m$ (for zero bias) or $p \leq 2^m - 1$ (for non-zero bias). For $N = 2^m + r$, $p \leq 2^m$ (for zero bias) or $p \leq 2^m - 1$ (for non-zero bias), m being any positive integer and r any positive integer $< 2^m$, a highly efficient design is represented by the matrix X obtained from the corresponding matrix X for the general design in Section 3 above by adding r rows 1, 1, $\cdots$ 1 to it. The matrix $X'X$ for these designs then comes out as

$$X'X = \begin{pmatrix} N & r & r & \cdots & r \\ r & N & r & \cdots & r \\ r & r & N & \cdots & r \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ r & r & r & \cdots & N \end{pmatrix} \tag{19}$$

which is of order $p \times p$ for zero bias, or of order $(p + 1) \times (p + 1)$ for non-zero bias. The variance of each unknown determined by this experiment is

$$\frac{\sigma^2}{N - \dfrac{(p-1)r^2}{N + (p-2)r}} \quad \text{for zero bias,} \tag{20}$$

or

$$\frac{\sigma^2}{N - \dfrac{pr^2}{N + (p-1)r}} \quad \text{if there is bias.} \tag{21}$$

Hence the efficiency of this design is

$$1 - \frac{(p-1)r^2}{N[N + (p-2)r]} \quad \text{for zero bias,} \tag{22}$$

or

$$1 - \frac{pr^2}{N[N + (p-1)r]} \quad \text{if there is bias.} \tag{23}$$

The loss of efficiency as a result of adopting this design is, therefore, $\dfrac{(p-1)r^2}{N[N + (p-2)r]}$ for zero bias, or $\dfrac{pr^2}{N[N + (p-1)r]}$ if there is bias

7. Second design for $N = 2^m + r$, $p \leq 2^m$ (for zero bias) or $p \leq 2^m - 1$ (for non-zero bias). Another design for these values of N and p is that represented by the matrix X obtained from the corresponding matrix X for the general design in Section 3 above by adding to it r rows $1, 0, 0, \cdots 0$. The matrix $X'X$ for this design is then given by

$$X'X = \begin{pmatrix} N & 0 & 0 & \cdots & 0 \\ 0 & N - r & 0 & \cdots & 0 \\ 0 & 0 & N - r & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & N - r \end{pmatrix}, \tag{24}$$

which is of order $p \times p$ if there is zero bias, or of order $(p + 1) \times (p + 1)$ if there is bias. Here also the estimates of all the unknowns are mutually orthogonal. The efficiency of the design comes out to be

$$\frac{(N - r)p}{Np - r} \quad \text{if there is zero bias,} \tag{25}$$

or

$$\frac{N - r}{N} \quad \text{if there is bias.} \tag{26}$$

By comparing the efficiency of the first design of this type given in (22) and (23) with that of the present design given in (25) and (26) respectively, it would appear that in case of zero bias, the efficiency of the first design is higher than that of the second design for $p > 1$, but equal for $p = 1$; and in case of non-zero bias, the efficiency of the first design is always higher than that of the second.

8. Comprehensive general design when N is at our choice. When N is at our choice, we can always obtain a completely orthogonalized design by taking N equal to a sufficiently large power of 2. For $p = 2^m$, m being any positive integer, a completely orthogonalized design for $N = 2^m$, when there is zero bias, has been given in Section 3 above. If, however, there is a bias, a completely orthogonalized design can be constructed for $N = 2^{m+1}$. When $p = 2^m + u$, where u is a positive integer $< 2^m$, a completely orthogonalized design is available for $N = 2^{m+1}$, whether the bias is zero or not.

For $N = 2^{m+1}$, this is the most efficient design, with 100 per cent efficiency, but as N is given higher powers of 2 than 2^{m+1}, the variance of the estimate of each unknown decreases. When $N = 2^l$, where $l > m + 1$, the variance of each unknown is $\frac{1}{2^{l-m-1}}$ of that for $N = 2^{m+1}$.

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NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

NOTE ON THE LAW OF LARGE NUMBERS AND "FAIR" GAMES

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1. "Fair" games. Let $\{X_k\}$ be a sequence of independent random variables with the same cumulative distribution function $V(x)$. Suppose that the expectation

$$E(X_k) = \int_{-\infty}^{+\infty} x\,dV(x) = M \tag{1}$$

exists, and put

$$S_n = X_1 + \cdots + X_n . \tag{2}$$

The weak law of large numbers states[1] that for every $\epsilon > 0$ and $n \to \infty$

$$\Pr\{| S_n - nM | < \epsilon n\} \to 1. \tag{3}$$

In the picturesque language of the theory of games this means that, after a large number of trials, the accumulated gain S_n will, with great probability, be of the order of magnitude of nM. This led to the definition that a game is "fair" if the entrance fee for each trial is M. Unfortunately this definition creates the erroneous notion that a "fair" game is necessarily fair. To disprove it we shall (section 3) exhibit an example which will show:

(I): *A game can be "fair" and nevertheless such that the probability tends to one that, after n trials, the player will have sustained a loss $L_n = nM - S_n$ of the order of magnitude $n(\log n)^{-\eta}$, where $\eta > 0$ is arbitrarily small. In other words, in our example*

$$\Pr\{nM - S_n > (1 - \epsilon)n(\log n)^{-\eta}\} \to 1. \tag{4}$$

Of course, L_n is necessarily of smaller order of magnitude than n; however, our example can be modified in such a way that the ratio of the loss L_n to the accumulated entrance fees nM decreases as slowly as one pleases.

This shows that a "fair" game can be exceedingly disadvantageous. Conversely, an "unfair" game can very well be advantageous. If a careful driver insures his car, the game is clearly "unfair" according to definition, and yet some

[1] Usually (3) is proved only under more restrictive hypotheses. Actually the finiteness of $E(X_k)$ implies even the strong law of large numbers; cf. Kolmogoroff, *Grundbegriffe der Wahrscheinlichkeitsrechnung* (Berlin 1933), p. 59.

states impose such games on drivers. Now in this and many other practical cases the game is of such a nature that there is a very small probability p of winning a comparatively great amount A; the "fair" price would be pA. In such cases the law of large numbers would be significant only if n is large compared to $1/p$, whereas actually the maximum number of games to be played is comparatively small. Clearly any theory meets practical requirements only if it makes allowance for the number of trials and makes the "fair" price depend on the number of trials.

2. The Petersburg "paradox." For obvious reasons the classical theory of probability was unable to provide a precise formulation of the law of large numbers and to establish the actual conditions of its validity. Often it has been looked upon as a direct consequence of the definition of probability, and this led to the so-called Petersburg paradox which presents no difficulties to the modern theory. It refers to the case where the expectation (1) is infinite. The usual example exhibits a game in which the possible gains in each trial are distributed according to

$$\Pr\{X = 2^k\} = 2^{-k} \tag{5}$$

Here $M = \infty$. Now the law of large numbers (3) used to be proved (if at all) only assuming the existence of moments of higher order. Nevertheless, the classical theory postulated the validity of (3) even for $M = \infty$, and treating ∞ as a number (with $\infty - \infty = 0$) it argued that ∞ is a "fair" price for the game as defined in (5). Great ingenuity was exercized in order to reconcile this result with commonsense.[2] Actually one can pass from (3) to the limit $M \to \infty$, but the *only* result to be arrived at is trivial and could be anticipated without theory: If the player pays for each trial a *fixed* amount A, he is likely to have a positive gain provided he plays sufficiently long, i.e., provided $n > N(A)$, where $N(A)$ itself *increases with* A

Instead of a paradox we reach the conclusion that the price should depend on n, that is to say vary as the number of trials increases. For best results this should be the case even if M is finite. It should be noticed that in the Petersburg case (5) a variable price can be determined so that a law of large numbers will hold which is in every respect analogous to (3). In this formula nM is simply the accumulated amount of entrance fees; denoting it by P_n, formula (3) takes on the equivalent form

[2] Among the latest textbooks, von Mises (*Wahrscheinlichkeitsrechnung*, Leipzig–Wien 1931, p. 108f.) avoids the difficulty by declaring that (5) can not represent a collectif because of its infinite tail. This viewpoint is legitimate, but makes the law of large numbers inapplicable to practically all useful distributions Fry (*Probability and its Engineering Uses*, New York, 1928, p. 197) says: "The true explanation of the paradox is . . based upon the fact that in our every-day experience we have to deal only with individuals who have finite fortunes and who would therefore be incapable of paying back the sums which are required . ." The problem does not seem to be mentioned in Uspensky's book

$$\Pr\{|S_n - P_n| < \epsilon P_n\} \to 1. \tag{6}$$

It is this interpretation of (3) that leads to the notion of "fair" games Now the Petersburg game can also be played in a "fair" way:

(II) *Let the player in the Petersburg game (5) at the k-th trial pay the amount*[3] $\log_2 k$. *The accumulated entrance fees up to the n-th trial are* $P_n \sim n \log_2 n$, *and the game is "fair" in the sense that the law of large numbers (6) holds This requirement determines the entrance fees essentially uniquely* (that is to say up to terms of smaller order of magnitude which, by definition, remain undetermined)

3. Proofs. Theorems (I) and (II) follow easily from the following

LEMMA: *Let* $a_n \to \infty$ *be a sequence of positive numbers; in order that there exist a sequence* $\{b_n\}$ *such that*

$$\Pr\{|S_n - b_n| < \epsilon a_n\} \to 1 \tag{7}$$

it is necessary and sufficient that for every $\delta > 0$ *simultaneously*

$$n \int_{|x| > \delta a_n} dV(x) \to 0, \qquad a_n^{-2} n \int_{|x| < a_n} x^2\, dV(x) \to 0. \tag{8}$$

in this case (8) *will hold with*

$$b_n = \sum_{k=1}^{n} \int_{|x| < a_k} x\, dV(x) \tag{9}$$

(*and, of course, for any other sequence* $\{b_n^*\}$ *if and only if* $|b_n^* - b_n| = O(a_n)$). This lemma is a simple consequence of the necessary and sufficient conditions for the generalized law of large numbers[4].

To prove theorem (II) we have to determine a sequence $\{a_n\}$ such that (7) will hold for the distribution function defined in (5) and with $b_n \sim a_n$. A simple computation shows that (8) will hold for any sequence $\{a_n\}$ which increases faster than n. Moreover, the sequence $\{b_n\}$ defined by (9) will be of the same order of magnitude as $\{a_n\}$ if, and only if, $a_n \sim n \log_2 n$. This proves (II)

Now let $\eta > 0$ be arbitrary, and define the distribution function $V(x)$ to have a density

$$V'(x) = \frac{\eta}{x^2 \log^{1+\eta} x} \qquad \text{for } x > e; \tag{10}$$

at $x = 0$ the function $V(x)$ shall have a jump of magnitude

$$1 - \int_e^\infty \frac{\eta\, dx}{x^2 \log^{1+\eta} x} < 1, \tag{11}$$

while $V(x)$ is constant in the intervals $x < 0$ and $0 < x < e$. For this distribution function we have obviously $M = 1$.

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[3] $\log_2$ stands for the logarithm to the basis 2

[4] Cf. FELLER, *Acta Univ. Szeged*, Vol. 8 (1937), pp. 191–201

Next, let for $n > e$

$$a_n = n \log^{-\eta} n. \tag{12}$$

Then (8) holds and from (9) and (10) we obtain easily for large n

$$b_n = \sum_{k=1}^{n} \{1 - \log^{-\eta} a_k\} < n - (1 - \epsilon)a_n . \tag{13}$$

Substituting into (7) one sees that, again for sufficiently large n,

$$\Pr \{S_n - n + (1 - \epsilon)a_n < \epsilon a_n\} \to 1, \tag{14}$$

or, since $M = 1$,

$$\Pr \{S_n - nM < -(1 - 2\epsilon)a_n\} \to 1. \tag{15}$$

This proves (I).

A NOTE ON RANK, MULTICOLLINEARITY AND MULTIPLE REGRESSION[1]

By Gerhard Tintner

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Let $X_{it}(i = 1, 2 \cdots M)$ be set of M random variables, each being observed at $t = 1, 2 \cdots N$. $X_{it} = M_{it} + y_{it}$. (This is essentially the situation envisaged by Frisch [1]). The systematic part of our variables $M_{it} = EX_{it}$. The y_{it} are normally distributed with means zero. Their variances and covariances are independent of t. The M_{it} and y_{it} are independent of each other Define $\bar{X}_i = \Sigma_t X_{it}/N$ the arithmetic mean of X_{it} and $x_{it} = X_{it} - \bar{X}_i$ the deviation from the mean. Then $a_{ij} = \Sigma_t x_{it} x_{jt}/(N - 1)$ gives the variances and covariances of the observations. We want to determine the rank of the matrix of the variances and covariances of M_{it}.

Now assume that $||V_{ij}||$ is an estimate of the variance-covariance matrix of the error terms or "disturbances" y_{it}. The elements of this matrix are distributed according to the Wishart distribution and are independent of the M_{it}. They can be estimated as deviations from polynomial trends, as deviations from Fourier series, by the Variate Difference Method, etc. The estimates could also be based upon a priori knowledge if for instance the y_{it} are interpreted as errors of measurement. Assume that the estimate is based upon N' observations.

[1] The author is much obliged to Professors W. G. Cochran (Iowa State College), H. Hotelling (Columbia University), T. Koopmans (University of Chicago) and A. Wald (Columbia University) for advice and criticism with this paper He has also profited by reading the unpublished paper: "On the Validity of an Estimate from a Multiple Regression Equation" by F. V Waugh and R. C. Been which deals in part with a problem related to the one presented here

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Form the determinantal equation:

$$|a_{ij} - \lambda V_{ij}| = 0. \tag{1}$$

Apart from sampling fluctuations there should be r solutions $\lambda = 1$ of equation (1) if there are r independent linear relationships between the M_{it}. The rank of the variance-covariance matrix of M_{it} is then $M - r$. Following a suggestion of P. L. Hsu [2] made on the basis of the earlier work of R. A. Fisher [3] we form the test function

$$\Lambda_r = (N - 1)(\lambda_1 + \lambda_2 \cdots + \lambda_r), \tag{2}$$

where λ_1 is the smallest root of (1), λ_2 the next smallest, etc. Hence (2) is the sum of the r smallest roots of equation (1). The hypothesis to be tested is that there are exactly r independent linear relationships between the systematic parts of our variables in the population. This quantity (2) is distributed like χ^2 with $r(N - M - 1 + r)$ degrees of freedom for large samples, i.e. if N' becomes large. It can be used for forming an opinion about the number of independent relationships existing among the systematic parts of our variables (M_{it}).

The importance of the question of the rank lies in the following: Sometimes we are not so much interested in making predictions as to estimate the "true" relationships which exist in the population which corresponds to our sample (Wald) [4]. Practically speaking, these relationships and their estimation are of great importance in economic statistics, as Haavelmo has shown [5]. But a knowledge of the rank i.e. the number of independent relationships existing between the systematic parts of the variables may also be of some significance for the problem of prediction. The inclusion of strongly correlated predictors cuts down on the number of degrees of freedom without contributing significantly to the reduction of the variance.

The remainder of this paper will be concerned with an attempt to estimate the relationships which in the population exist between the systematic parts of the variables. This is an extension of the work of T. Koopmans [6] and the author [7] who dealt with the special case in which there is only one relationship between the systematic parts.

Suppose that we decide that there are R independent relationships among the systematic parts of our variables

$$k_{v0} + \sum_j k_{vj} M_{jt} = f_{vt} = 0; \qquad v = 1, 2, \cdots, R, t = 1, 2, \cdots, N. \tag{3}$$

We desire to obtain estimates of these relationships. Our purpose here is not prediction but estimation of the structural coefficients k_{vj}.

The method of maximum likelihood leads to the method of least squares if we treat the V_{ij} as constants. This is again permissible if N' is large and our estimates of the V_{ij} become reasonably accurate. We have to minimize the following sum of squares

$$Q = \sum_t Q_t \tag{4}$$

where

$$Q_t = \sum_i \sum_j V^{ij}(x_{it} - m_{it})(x_{jt} - m_{jt}), \tag{5}$$

where $\| V^{ij} \| = \| V_{ij} \|^{-1}$, the inverse of the variance-covariance matrix of the errors. We also define $m_{it} = M_{it} - \bar{M}_i$, $(t = 1, 2, \cdots, N)$ where $\bar{M}_i$ is the mean of M_{it}.

If there are R relationships (3) they can be written by using only $R(M - R)$ coefficients $k_{vj}(j = 1, 2 \cdots M)$, if we disregard the constant terms k_{0v}, because we are now dealing with deviations from means. We can for instance express the first $(M - R)$ variables m_{it} in terms of the last R variables m_{it}. Hence, we have to impose R^2 conditions upon the MR coefficients $k_{vj}(j = 1, 2, \cdots, M)$ appearing in (3).

We impose $R(R + 1)/2$ conditions as follows

$$\Sigma_i \Sigma_j k_{vi} k_{wj} V_{ij} = g_{vw} = \delta_{vw}, \tag{6}$$

where δ_{vw} is a Kronecker delta. These conditions orthogonalize and normalize the coefficients k_{vj}. We have now to adjust the Q_t as given in (5) under the conditions (6) by determining appropriate m_{it}. This is a problem of restricted minima.

We introduce a new function

$$F_t = Q_t - \sum_v \mu_{vt} f_{vt}, \tag{7}$$

where the μ_{vt} are Lagrange multipliers. Differentiating with respect to m_{it} and setting equal to zero we get the solution:

$$\sum_j V^{ij}(x_{jt} - m_{jt}) = \sum_v \mu_{vt} k_{vi}; \qquad (i = 1, 2, \cdots, M); \tag{8}$$

or, solving for $x_{it} - m_{it}$

$$x_{it} - m_{it} = \sum_v \sum_j \mu_{vt} V_{ij} k_{vj}; \qquad i = 1, 2, \cdots, M. \tag{9}$$

Multiplying (9) by k_{vi} and summing we get

$$\mu_{vt} = \Sigma_j k_{vj} x_{jt}. \tag{10}$$

Hence we have

$$Q_t = \sum_v \mu_{vt}^2 = \sum_v \left(\sum_j k_{vj} x_{jt} \right)^2. \tag{11}$$

Now we dispose of the remaining $R(R - 1)/2$ conditions

$$\sum_t \mu_{vt} \mu_{wt} = h_{vw} = 0, \qquad v \neq w. \tag{12}$$

We have to maximize Q under the R^2 conditions (6) and (12). This is done by finding the appropriate k_{vj}.

We form a new expression

$$G = Q + \Sigma_v \Sigma_w \beta_{vw} h_{vw} - \Sigma_v \Sigma_w \alpha_{vw} g_{vw} \tag{13}$$

where the α_{vw} and β_{vw} $(v \neq w)$ are again Lagrange multipliers and $\beta_{vv} = 0$. Because of considerations of symmetry we have: $\alpha_{vw} = \alpha_{wv}$ and $\beta_{vw} = \beta_{wv}$ Differentiating with respect to k_{vi} and setting equal to zero we get the condition

$$\begin{aligned} &\Sigma_t(\Sigma_j k_{vj} x_{jt}) x_{it} + \Sigma_w \beta_{vw} \Sigma_t(\Sigma_j k_{wj} x_{jt}) x_{it} \\ &\quad = \Sigma_w \alpha_{vw} \Sigma_j V_{ij} k_{wj}, \qquad v = 1, 2, \cdots, R, \qquad i = 1, 2, \cdots, M. \end{aligned} \tag{14}$$

Multiplying by k_{vi} and summing we get

$$\Sigma_t \mu_{vt}^2 = \alpha_{vv} \tag{15}$$

Multiplying by k_{zi} $(z \neq v)$ and summing we have

$$\beta_{vz} \Sigma_t \mu_{zt}^2 = \alpha_{vz} \qquad (v \neq z). \tag{16}$$

Both (15) and (16) follow from conditions (6) and (12)

Exchanging the role of v and z in (16) we have also

$$\beta_{vz} \Sigma_t \mu_{vt}^2 = \alpha_{vz} \qquad (v \neq z). \tag{17}$$

Hence we have $\alpha_{vz} = \beta_{vz} = 0$, if $v \neq w$ Inserting these results in (14) we get a system of linear and homogeneous equations in the unknown coefficients k_{vj}. The determinant of the system must be equal to zero in order to yield non-trivial solutions. Trivial solutions are not admitted because of (6) Hence the α_{vv} are simply the roots k of the equation $| \Sigma_t x_{it} x_{jt} - k V_{ij} | = 0$.

Introducing

$$\lambda_v = \alpha_{vv}/(N - 1), \tag{18}$$

expression (14) becomes actually the determinantal equation (1). This expression can be used to find the R smallest latent roots λ_v and the corresponding characteristic vectors k_{vj} by Hotelling's methods [8]

The constants of the equation (3) are finally determined by the condition that the optimum solutions have to go through the means of the variables

$$k_{v0} + \Sigma_j k_{vj} \bar{X}_j = 0. \tag{19}$$

The distribution of the variances and covariances of the observations has recently been established by T. W. Anderson and M. A Girshick for the cases $R = M - 1$ and $R = M - 2$ [9]

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NOTE ON THE DISTRIBUTION OF THE SERIAL CORRELATION COEFFICIENT[1]

BY WILLIAM G. MADOW

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The distribution of the serial correlation coefficient when $\rho = 0$ has been previously obtained.[2] The purpose of this note is to derive the distribution of the serial correlation coefficient, using the circular definition, when $\rho \neq 0$.

Let us assume that the random variables $x_1, \cdots, x_N$ have a joint normal distribution[3] $p(x_1, \cdots, x_N \mid A, B, \mu)$ where

$$\log p(x_1, \cdots, x_N \mid A, B, \mu) = \log K_1 - \tfrac{1}{2}\left[A \sum_i (x_i - \mu)^2 + 2B \sum_i (x_i - \mu)(x_{i+L} - \mu)\right]$$

the term in the bracket is positive definite, K_1 is independent of the x_i and if $i + L > N$ then $x_{i+L} = x_{i+L-N}$. It is then clear that $\bar{x}$, V_N, and ${}_LC_N$, where $\bar{x}$ is the arithmetic mean, $V_N = \sum_i (x_i - \bar{x})^2$ and

$${}_LC_N = \sum_i (x_i - \bar{x})(x_{i+L} - \bar{x})$$

are sufficient statistics with respect to the estimation of μ, A, and B.

Let $V_N\, {}_LR_N = {}_LC_N$ define ${}_LR_N$, the serial correlation coefficient. Then if

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[1] Presented at a meeting of the Cowles Commission for Economic Research in Chicago, January 31, 1945.

[2] See R. L. Anderson, "*Distribution of the serial correlation coefficient*", pp. 1–13 and T. Koopmans, "Serial correlation and quadratic forms in normal variables", pp. 14–33, *Annals of Math. Stat.*, Vol. XIII, No 1, March, 1942.

[3] The expression $p(\xi_1, \cdots, \xi_m \mid \theta_1, \cdots, \theta_g)$ means the probability density or the distribution of the random variables $\xi_1, \cdots, \xi_m$ for the given values of the parameters $\theta_1, \cdots, \theta_g$. When used as an index of summation or multiplication, the letter i will assume all values from 1 through N.

$A = 1, B = 0$ Anderson has shown[4] that, if N is odd, the joint distribution of ${}_1R_N$ and V_N is given by

$$(1)\quad D(R_N, V_N) = KV_N^{\frac{1}{2}(N-3)} e^{-\frac{1}{2}V_N} \sum_{i=1}^{m} (\lambda_i - R_N)^{\frac{1}{2}(N-5)}/\alpha_i, \quad \text{for } \lambda_{m+1} \leq R_N \leq \lambda_m$$

where

$$R_N = {}_1R_N, \qquad \lambda_k = \cos\frac{2\pi k}{N}, \alpha_i = \prod_{j=1}^{\frac{1}{2}(N-1)} (\lambda_i - \lambda_j); \qquad \text{for all } j \neq i$$

and $K^{-1} = 2^{\frac{1}{2}(N-1)}\,\Gamma[\frac{1}{2}(N-3)]$; while if N is even, the same formula holds except that

$$\alpha_i = \prod_{j=1}^{\frac{1}{2}(N-2)} (\lambda_i - \lambda_j)\sqrt{(\lambda_i + 1)}, \qquad \text{for all } j \neq i.$$

We now extend Anderson's distributions to the case where it is not assumed that $A = 1$ and $B = 0$.

As a means of extending[5] Anderson's distribution let us recall that if $x_1, \cdots, x_N$ have a distribution $p(x_1, \cdots, x_N \mid \theta_1, \cdots, \theta_g)$ depending on several parameters $\theta_1, \cdots, \theta_g$, and if $z_1, \cdots, z_k$ are a sufficient set of statistics with respect to $\theta_1, \cdots, \theta_g$, i.e.

$$p(x\ , \cdots, x_N \mid \theta_1, \cdots, \theta_g) = h(z_1, \cdots, z_k \mid \theta_1, \cdots, \theta_g)m(x_1, \cdots, x_N)$$

where $m(x_1, \cdots, x_N)$ is independent of $\theta_1, \cdots, \theta_g$, then if the distribution of $z_1, \cdots, z_k$ is found, assuming $\theta_1, \cdots, \theta_g$ have specific values $\theta_1^0, \cdots, \theta_g^0$, then it follows that

$$p(z_1, \cdots, z_k \mid \theta_1, \cdots, \theta_g) = p(z_1, \cdots, z_k \mid \theta_1^0, \cdots \theta_g^0)\frac{h(z_1, \cdots, z_k \mid \theta_1, \cdots, \theta_g)}{h(z_1, \cdots, z_k \mid \theta_1^0, \cdots, \theta_g^0)}.$$

We may call Anderson's distribution given in (1), $p(R_N, V_N \mid 1, 0)$, i.e.

$$p(R_N, V_N \mid 1, 0) \quad = \quad D(R_N, V_N)$$

Furthermore, $\bar{x}$ is distributed independently of R_N and V_N for all values of A and B and hence by a simple transformation,[6] we can apply the above theorem.

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[4] Anderson loc. cit p. 3 and p. 5. Although the remainder of the note deals only with the case where $L = 1$ the procedure is general and may be easily carried through for other lags

[5] See W. G. Madow Contributions to the "Theory of multivariate statistical analysis", *Trans. of the Amer. Math Soc.*, Vol. 44, No. 3, November 1938, p. 461.

[6] For a proof that an orthogonal transformation of the variable $x_i - \mu$ exists such that V_N and ${}_LC_N$ are simultaneously reduced to canonical forms involving the same $N - 1$ of the variables of the transformation, and $\sqrt{N}\,(\bar{x} - \mu)$ is the Nth variable of the transformation, see J. von Neumann, "Distribution of the ratio of the mean square successive difference to the variance, *Annals of Math. Stat*, Vol. XII, No. 4, December 1941, pp 368, 369. The proof there is given for V_N and $\sum_i (x_i - x_{i+1})^2$ but is easily extended to this case

Then it is easy to show that $N(\bar{x} - \mu)$ is independently distributed of V_N, and ${}_LC_N$ and has distribution $\log p[\sqrt{N}(\bar{x} - \mu) \mid A, B] = \log K_2 - \frac{1}{2}[A + 2B]N(\bar{x} - \mu)_2$ where $K_2 = (2\pi)^{-\frac{1}{2}}(A + 2B)^{\frac{1}{2}}$ and $K_1'K_2 = K_1$

Then

$$p(R_N, V_N \mid A, B) = p(R_N, V_N \mid 1, 0)\Omega$$

where

$$\Omega = \frac{K_1' e^{-\frac{1}{2}(AV_N+2BR_NV_N)}}{(2\pi)^{-\frac{1}{2}N} e^{-\frac{1}{2}V_N}}.$$

Hence it follows that,

$$p(R_N, V_N \mid A, B) = KK_1'(2\pi)^{\frac{1}{2}N} V_N^{\frac{1}{2}(N-3)} e^{-\frac{1}{2}V_N(A+2BR_N)} \sum_{i=1}^{m} (\lambda_i - R_N)^{\frac{1}{2}(N-5)}/\alpha_i,$$

for $\lambda_{m+1} \leq R_N \leq \lambda_m$, where the α_i have different values according to whether N is odd or even. In order to evaluate $p(R_N \mid A, B)$ we then need only integrate out V_N. Now

$$\int_0^\infty V_N^{\frac{1}{2}(N-3)} e^{-\frac{1}{2}V_N(A+2BR_N)} dV_N = \Gamma[\tfrac{1}{2}(N-1)](A/2 + BR_N)^{-\frac{1}{2}(N-1)}.$$

Hence

$$p(R_N \mid A, B) = KK_1'(2\pi)^{\frac{1}{2}N}\Gamma[\tfrac{1}{2}(N-1)](A/2 + BR_N)^{-\frac{1}{2}(N-1)} \sum_{i=1}^{m} (\lambda_i - R_N)^{\frac{1}{2}(N-5)}/\alpha_i.$$

The parameters K_1', A and B depend on the different types of assumptions that may be made. In general

$$K_1 = (2\pi)^{-\frac{1}{2}N}\Delta^{1/2}$$

where Δ is a circulant $(a_1, \cdots, a_N)$ such that

$$a_1 = A, \qquad a_{1+L} = B, \qquad a_{1+(N-L)} = B, \qquad a_i = 0 \text{ otherwise},$$

and hence

$$\Delta = \prod_i \left(A + B\cos\frac{2\pi iL}{N}\right) = \prod (A + B\lambda_i).$$

Then, one assumption is

$$A = \frac{1}{\sigma^2}, \qquad B = -\rho/\sigma^2$$

where ρ is the "true" serial correlation coefficient. Other assumptions are possible.[7] However, these vary with the problem under consideration and may be left for further examination.

[7] One possible alternative definition is given by W. J. Dixon, "Further contributions to the problem of serial correlation", *Annals of Math. Stat.*, Vol. XV, No. 2, June 1944, p. 120, equation (2.1).

NOTE ON A PAPER BY C. W. COTTERMAN AND L. H. SNYDER

By H. B. Mann[1]

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C. W. Cotterman and L. H. Snyder [1] gave a method to test simple Mendelian inheritance in randomly collected data From a population assumed to be at equilibrium a sample is taken The number of homozygous recessives in the sample is known We wish to estimate the number of heterozygous individuals in the sample.

Let α be the proportion of recessive genes among all genes in the population; π, ρ, τ the proportion in the population of homozygous recessives, heterozygous and homozygous dominant individuals respectively and p, r, t the sampling values of π, ρ, τ. Then

$$\pi = \alpha^2, \rho = 2\alpha(1 - \alpha), \tau = (1 - \alpha)^2, p + r + t = 1. \tag{1}$$

Cotterman and Snyder use as an estimate of r the quantity $2\sqrt{p}(1 - \sqrt{p})$. It is the purpose of this note to show that this estimate is for all practical purposes equivalent to the maximum likelihood estimate of r.

The joint distribution of p, r and t in samples of n is given by

$$P(p, r, t) = \frac{n!\pi^{np}\rho^{nr}\tau^{nt}}{(np)!(nr)!(nt)!} = \frac{n!\alpha^{2np}[2\alpha(1 - \alpha)]^{nr}(1 - \alpha)^{2nt}}{(np)!(nr)!(nt)!}, \tag{2}$$

where $P(p, r, t)$ is the probability of obtaining the values p, r, t in samples of n.

We wish to maximize $P(p, r, t)$ for fixed values of p with respect to α and r. Maximizing first with respect to α one easily obtains

$$2\alpha = 2p + r. \tag{3}$$

We can regard α as a continuous parameter and hence (3) must hold at any maximum of $P(p, r, t)$. For any maximum of $P(p, r, t)$ we must further have

$$\frac{n!\pi^{np}\rho^{nr}\tau^{nt}}{(np)!(nr)!(nt)!} \geqslant \frac{n!\pi^{np}\rho^{nr+1}\tau^{nt-1}}{(np)!(nr + 1)!(nt - 1)!}$$

and

$$\frac{n!\pi^{np}\rho^{nr}\tau^{nt}}{(np)!(nr)!(nt)!} \geqslant \frac{n!\pi^{np}\rho^{nr-1}\tau^{nt-1}}{(np)!(nr - 1)!(nt + 1)!}.$$

This leads to the inequalities

$$\frac{\tau}{nt} \geqslant \frac{\rho}{nr + 1}, \qquad \frac{\rho}{nr} \geqslant \frac{\tau}{nt + 1}. \tag{4}$$

Substituting $t = 1 - p - r$, $\tau = 1 - \pi - \rho$ one easily obtains from (4)

[1] Research under a grant of the research foundation of Ohio State University.

$$\frac{\rho n - \rho n p + \rho}{n(1-\pi)} \geqslant r \geqslant \frac{\rho n - \rho n p - \tau}{n(1-\pi)}. \tag{5}$$

The difference of the two bounds is $\frac{1}{n}$. Hence r must satisfy an equation

$$r = \frac{\rho n - \rho n p + \rho}{n(1-\pi)} - \frac{\epsilon}{n}, \qquad 0 \leqslant \epsilon \leqslant 1.$$

Substituting the values for ρ, π and r from (1) and (3) we obtain

$$\alpha^2 - \frac{\alpha}{n}(1 - \epsilon/2) - p + \frac{\epsilon}{2n} = 0,$$

$$\alpha = \frac{2-\epsilon}{4n} + \frac{1}{2}\sqrt{\frac{(2-\epsilon)^2}{4n^2} + 4p - \frac{2\epsilon}{n}}.$$

Since $0 \leqslant \epsilon \leqslant 1$ we obtain from (3)

$$\frac{1}{n} + \sqrt{4p + \frac{1}{n^2}} - 2p \geqslant r \geqslant \frac{1}{2n} + \sqrt{4p + \frac{1}{4n^2} - \frac{2}{n}} - 2p \tag{6}$$

From (6) we see that for all practical purposes we may use the estimate

$$r = 2\sqrt{p}(1 - \sqrt{p}).$$

REFERENCE

[1] C. W. Cotterman and L.H. Snyder, "Tests of simple Mendelian inheritance in randomly collected data of one and two generations," *Jour. Am. Stat. Assn.*, Vol. 34 (1939), pp. 511-523.

NEWS AND NOTICES

Readers are invited to submit to the Secretary of the Institute new items of interest

Personal Items

Dr. R G. D. Allen, who has been associated with the Combined Production and Resources in Washington has returned to the London School of Economics.

Dr. Kenneth J. Arnold, who has been doing war research work with the Columbia University Statistical Research Group has returned to his position at the University of Wisconsin.

Dr. Lee A. Aroian, on leave from Hunter College is serving as a research associate in the Applied Mathematics Panel Project at Berkeley, California under the direction of Professor Neyman.

Dr. Ernest E. Blanche, has been appointed to the teaching staff of the Army University organized by the War Department for American veterans at Florence, Italy.

Assistant Professor Z. W. Birnbaum of the University of Washington has been promoted to an associate professorship.

Dr. Alva E. Brandt has returned from the Operational Research Section of the Ninth Air Force in Europe.

Associate Professor R. S. Burington of the Case School of Applied Science has received the Meritorious Civilian Award from the United States Navy.

Dr. Irving W. Burr has been promoted to an associate professorship at Purdue University.

Miss Frances Campbell, after receiving her doctorate at Michigan in June, has returned to her position at George Pepperdine College, Los Angeles.

Professor Harry C. Carver, after a year of service with the Army Air Forces, has returned to the University of Michigan.

Professor W. G. Cochran has returned to Iowa State College from a special mission to Germany.

Professor Churchill Eisenhart, who has been doing war research work with the Columbia University Statistical Research Group, has returned to the University of Wisconsin.

Miss Mary Elveback has been appointed to an assistant professorship at Rockford College.

Assistant Professor C. H. Fischer of the University of Michigan has been promoted to an associate professorship.

Mr. Elvin A. Hoy, who has spent three years with the War Production Board, is now Chief of the Statistics Section of the Bureau of Research and Statistics of the Social Security Board.

Professor P. L. Hsu of Kunming, China, has been appointed to a visiting professorship of statistics at Columbia University, beginning January 1946.

Dr. Doncaster G Humm has received an honorary Doctor of Science degree at Bucknell University

Mr. Joseph M. Juran who has served during the war with the Foreign Economic Administration, is now Chairman of the Department of Administrative Engineering at New York University

Dr Eugene Lukacs has been appointed Professor and Head of the Mathematics Department at Our Lady of Cincinnati College.

Dr. R. v. Mises of Harvard University has been appointed to a professorship of aerodynamics and applied mathematics.

Professor A M. Mood has returned from Princeton University to his position at Iowa State College.

Assistant Professor Henry Scheffé of Syracuse University has been granted leave of absence to serve as senior mathematician with Princeton University Station of Division 2 of NDRC

Symposium at the University of California

A Symposium on Mathematical Statistics and Probability was held at the University of California at Berkeley on August 13–18, 1945. Those participating in the symposium as speakers or chairmen were:

Dean G P Adams, Prof E. B Babcock, Prof. E M. Beesley, Prof B A Beinstein, Prof. Egon Brunswik, Prof. A H Copeland, Prof P H. Daus, Lt. Comm F W Dresch, Prof G. C Evans, Miss Evelyn Fix, Prof Harold Hotelling, Prof Victor F. Lenzen, Prof Jay L. Lush, Prof. J. H McDonald, Prof George F. McEwen, Prof J. Neyman, Prof G Polya, Prof. Hans Reichenbach, Prof A C. Schaeffer, Prof. Morgan Ward, and Dr Jacob Wolfowitz.

New Members

The following persons have been elected to membership in the Institute·

Abbey, Helen, M.A. (Michigan) Stat, Bur. of Records & Stat. Mich. Dept of Health, *916 N Chestnut, Lansing, Michigan.*

Acton, Forman, Ch. E. (Princeton) *T/4 Army of the U S, SED Barracks Area, Oak Ridge, Tenn.*

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Blackwell, Asst. Prof. David, Ph.D (Illinois) *Math Dept. Howard Univ. Wash, D. C*

Borland, James, M.A. (Indiana) Capt, Ex. Officer, *Inspect. Office, Pine Bluff Arsenal, Ark*

Brown, Prof. Theo., Ph.D (Yale) Bus. Stat. *Harvard Bus. School, Soldier's Field, Boston 63, Mass.*

Bunke, Alfred, M A (Columbia) Sen. Stat. *N Y. State Dept. of Labor, 37 Parkwood St. Albany 3, N. Y.*

Burington, Asso. Prof. Richard, Ph.D (Ohio) On leave from Case School of Applied Science, Cleveland, Ohio, at Present, Head Math, Bu. Ord USN *5200 N. Carlin Spring Rd, Arlington, Va.*

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Churchill, Edmund, A M (Columbia) *1685 Union Port Road, New York 2, N. Y.*

Cornfield, Jerome, B S (New York Univ.) *Stat. Dept of Labor, R F D #2 Herndon, Va*

Cruden, Dorothy, A B. (California) Stat in Sampling Sect *Spec. Sur Div Bur of Census % Pop. Div. Wash , D. C.*

Daniel, Cuthbert, M S (Mass Inst Tech) Stat Eng , Carbide and Carbon Chem. Corp , *460 East Drive, Oak Ridge, Tenn*

David, Florence, Ph D (London) Univ Sect Stat *Dept Univ Coll , London, W.C. 1, England*

De Garis, Prof. Charles, Ph D (Johns Hopkins) *Univ. of Okla School of Med , Okla. City, 4. Okla*

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Ede, Richard, B.S (Wisconsin) Chemistry Devel. Metallurgist, Gary Works, Car. Steel Ill. *547 Fillmore St , Gary, Indiana.*

Ewart, Robert, A B. (New York Univ.) Research Physicist, Ballistics Dept. Des Moines Ord. Plant *683-46th St. Des Moines 12, Iowa.*

Federer, Walter, M S (Kansas State) Research-Ag *Stat Stat Lab , Iowa State Coll. Ames, Iowa*

Freeman, Richard, B Sc. (McMaster) Research Chemist. *1 Maple Ave., Hamilton, Ontario, Canada*

Goldrosen, David, B S. (Worcester Poly Inst) Lt USNR Quality Control Officer, Insp. of Naval Mat'l *204 Ward St Newton Centre, Mass.*

Goodman, Albert, Supervisor Stat. Control, *Quality Control, Westinghouse Elec. Corp., Essington, Pa.*

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Nichols, Russell, B.A. (DePauw) *Sergeant, U. S. Army Co. A. 526 A 1, Kn APO 655, NYC (33-745-907).*

O'Neil, Frank (Lowell Textile Inst.) Senior Textile Technician, *Worsted Division, Pacific Mills', Lawrence, Mass.*

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Spoerl, Charles, B.A. (Harvard) Asst. Treas. *%Aetna Life Ins. Co. Hartford, Conn*

Springer, Wm., C.E. (Columbia) Asst Vice Pres in charge of Research, *Bristol-Myers Co. Hillside 5, New Jersey.*

Stock, J. Stevens, M.A. (American) Lt. USNR, Hd. Stat. Sect. Div. of Shore Est. & Civilian Per. *Navy Dept., 8508 Garfield St , Bethesda, Maryland.*

Stott, Alex, A.B. (Harvard) Lt. Comdr. USNR, *2800 Devonshire Pl., N.W , Wash 8, D. C*

Taylor, Thomas, Ph.D. (Yale) Research Engineer, U. S. Testing Co. *45 Grover Lane, Caldwell, N. J.*

Treanor, Glen, B.A. (Minnesota) Principal Tax Economist, Bus & Ind. Research Div., *Income Tax Unit, Bur. of Int. Rev., Room 2232, Wash , D. C.*

Wherry, Robert, Ph.D. (Ohio State) On leave Dept. Psych. Univ. of N. C., Civilian Head, Stat. Anal. Unit AGO Personnel Research Section, *270 Madison Ave , N. Y.*

Wilcoxon, Frank, Ph.D (Cornell) Group Leader, Insecticide & Fungicide, La , Amer. Cyanamid Co., Stamford, Conn. *R.D #1 Box 39a, Riverside, Conn.*

Wolff, Marion, A B. (Hunter) Asst. Math. Stat. Stat. Research Group Div. of War Research Columbia University *1724 Crotona Park East, New York 60, N Y.*

Unknown Addresses

Recent mail has not been delivered to the following members of the Institute at the addresses listed. If anyone knows of the current address of one or more of these members, please notify the Secretary-Treasurer at once.

Lt. (jg) Gordon L. Beckstead—Aer. Navy 151 % Fleet Postmaster, San Fran., Cal.
Dr. Charles Wm. Cotterman—637 Hawthorne Road, Winston Salem, North Carolina
Mr. James Davidson—Box 344, Christiansburg, Virginia
S/sgt George Elmstrom—Det of Pat., Hospital Plant. 4176 APO % PM, NYC, N. Y.
Mr. Henry Goldberg—401 W. 118th St. New York 27, New York
Mr Henry Hobley—Box 166, Pittsburgh 30, Pennsylvania
Mr. John Mandel—45 Kew Gardens Road, Kew Gardens, New York
Mr. David F. Votaw, Jr., USNTC—Bainbridge, Maryland
Mr. Edward F. Wilson—Keswick Colony, Keswick Grove, New Jersey

REPORT ON THE RUTGERS MEETING OF THE INSTITUTE

The Eighth Summer Meeting of the Institute of Mathematical Statistics was held at the New Jersey College for Women, Rutgers University, New Brunswick, New Jersey on Sunday, September 16, 1945, where the Summer Meeting of the American Mathematical Society was also being held. The following 115 members of the Institute attended the meeting:

C B. Allendoerfer, R. L Anderson, T. W. Anderson, H. E Arnold, I. L. Battin, Archie Blake, C. I. Bliss, P. Boschan, A. H Bowker, A. E. Brandt, G W. Brown, R H. Brown, T. H Brown, T A. Budne, R. S Burington, B. H. Camp, A. G. Carlton, P. C. Clifford, E. P. Coleman, T F. Cope, G M. Cox, H. B. Curry, J. H Curtiss, J. F Daly, J. H. Davidson, B B Day, W E. Deming, H F. Dodge, Jacques Dutka, P. S Dwyer, Churchill Eisenhart, Wade Ellis, Mary Elveback, Benjamin Epstein, C. D Ferris, C. H. Fischer, M. M. Flood, R M. Foster, Milton Friedman, J. P Gill, M. A. Girshick, Casper Goffman, A A Goodman, Dorothy K. Gottfried, T N. E. Greville, F. E. Grubbs, K W. Halbert, Marshall Hall, P. R Halmos, Miriam S. Harold, Millard Hastay, Bernard Hecht, William Hodgkinson, I. S. Hoffer, Harold Hotelling, A. S Householder, W. Hurwicz, Irving Kaplansky, C. J. Kirchen, Jack Laderman, Rafael Laguardia, H G. Landau, Howard Levene, Harriet Levine, S. B. Littauer, A. T. Lonseth, P J McCarthy, W G. Madow, J. W. Mauchly, E. B. Mode, D. J. Morrow, J. E Morton, Judith Moss, P. M Neurath, M. L Norden, H. W. Norton, C. O. Oakley, P S Olmstead, Edward Paulson, John Riodan, H. E Robbins, H G Romig, William Salkind, M. M. Sandomire, Arthur Sard, F E Satterthwaite, L. J. Savage, Henry Scheffé, Bernice Scherl, Edward Schrock, I. E. Segal, C. E Shannon, L. W Shaw, Herbert Solomon, Mortimer Spiegelman, J. R Steen, Arthur Stein, F. F. Stephan, A. P. Stergion, L. V. Toralballa, Mary N. Torrey, A. W. Tucker, L. R Tucker, J. W Tukey, Helen M. Walker, W. A. Wallis, R M. Walter, B T. Weber, Joseph Weinstein, A. E. R. Westman, Frank Wilcoxon, S S. Wilks, Jacob Wolfowitz, C. P Winsor, Ruth Zwerling

The first session, on Sunday morning, was devoted to a symposium on *Sequential Analysis*. Professor W. Allen Wallis, of Stanford University and Columbia Statistical Research Group, acted as chairman for this session. The following invited addresses were given.

1 *Theory of Sequential Analysis.*
   Professor A. Wald, Columbia University and Columbia Statistical Research Group.
2. *Construction of Multiple Sampling Inspection Plans for Attributes from Sequential Principles*
   Mr Milton Friedman, National Bureau of Economic Research and Columbia Statistical Research Group.
3 *Applications of Sequential Analysis to the Ranking of Two Populations with Respect to a Single Parameter.*
   Mr. M A Girshick, Bureau of Agricultural Economics and Columbia Statistical Research Group.

The morning session was concluded after lively discussion on the symposium topic.

Dr. W. Edwards Deming, of the Bureau of the Budget and President of the Institute, presided at the afternoon session The following papers were presented:

1. *On The Variance of a Random Set in n Dimensions.*
   Dr. Herbert E. Robbins, Post Graduate School, Annapolis.
2. *The Non-Central Wishart Distribution and its Application to Problems In Multivariate Statistics.*
   Dr. T. W. Anderson, Jr, Princeton University.
3. *The Effect on a Distribution Function of Small Changes in the Population Function.*
   Professor Burton H Camp, Wesleyan University
4. *On Composite Distributions.*
   Dr. Casper Goffman and Dr. Benjamin Epstein, Westinghouse Electric Corp.
5. *Population, Expected Values and Sample.*
   Professor Emil J. Gumbel, New School for Social Research
6. *On the Selection of a Sample in Repeated Steps.*
   Dr. W. G. Madow, Bureau of the Census.
7. *On Optimum Estimates for Stratified Samples.*
   Mr. Morris H. Hansen and Mr. William N. Hurwitz, Bureau of the Census Presented by Margaret Gurney.
8. *Pearsonian Correlation Coefficients Associated With Least Squares Theory* (Presented by Title).
   Professor P. S. Dwyer, University of Michigan.

The afternoon session concluded with the report of the Committee on the Teaching of Statistics which was presented by Professor Harold Hotelling of Columbia University.

P S. DWYER,
Secretary

ON THE NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION

By W Feller

Cornell University

1. Although the problem of an efficient estimation of the error in the normal approximation to the binomial distribution is classical, the many papers which are still being written on the subject show that not all pertinent questions have found a satisfactory solution. Let for a fixed n and $0 < p < 1$, $q = 1 - p$,

$$T_k = \binom{n}{k} p^k q^{n-k}, \qquad P_{\lambda,\nu} = \sum_{k=\lambda}^{\nu} T_k . \tag{1}$$

For reasons of tradition (and, apparently, only for such reasons) one sets

$$z_k = (k - np)\sigma^{-1}, \qquad \sigma = (npq)^{1/2}, \tag{2}$$

and compares (1) with

$$N_k = (2\pi)^{-1/2} \sigma^{-1} e^{-z_k^2/2} \quad \text{and} \quad \Pi_{\lambda,\nu} = \Phi\left(z_\nu + \frac{1}{2\sigma}\right) - \Phi\left(z_\lambda - \frac{1}{2\sigma}\right) \tag{3}$$

respectively,[1] where $\Phi(z)$ stands for the normalized error function. Many estimates are available for the maximum of the difference $| P_{\lambda,\nu} - \Pi_{\lambda,\nu} |$ for all λ, ν. Now this error is $O(\sigma^{-1})$ and even a precise appraisal will break down in the two most interesting cases: if σ is small, or if λ and ν are large as compared to σ. Indeed, even for moderately large values of k (such as are usually considered) the contribution of T_k to the sum in (1) will be considerably smaller than σ^{-1} so that any estimate of the form $O(\sigma^{-1})$ leaves us without guidance With some modifications this remains true also for more refined estimates like Uspensky's remarkable result[2]

$$P_{\lambda,\nu} = \Pi_{\lambda,\nu} + \frac{q - p}{6\sigma(2\pi)^{1/2}} [(1 - z^2)e^{-z^2/2}] \Big|_{z_\lambda - 1/2\sigma}^{z_\nu + 1/2\sigma} + \omega \tag{4}$$

with

$$| \omega | < \{.13 + .18 \, | p - q |\}\sigma^{-2} + e^{-3\sigma/2}$$

provided $\sigma \geq 5$ What is really needed in many applications is an estimate of the relative error, but this seems difficult to obtain.

It should also be noticed that the accuracy of the normal approximation to the binomial is by no means quite as good as many texts would make appear. Exam-

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[1] Very often the limits z_λ and z_ν instead of $z_\nu + \frac{1}{2\sigma}$ and $z_\lambda - \frac{1}{2\sigma}$ are used This naturally results in an unnecessary systematic undervaluation

[2] Uspensky [3], p 129 A two-term development of T_r with an error of $O(\sigma^{-2})$ valid for $| x | < 2$, $\sigma > 3$ has been given by Mirimanoff and Dovaz [1927]

ples using $p = \frac{1}{2}$ and intervals which are symmetric with respect to np are hardly conclusive, since there the main error term drops out and systematic positive and negative errors cancel. Again, in practice comparatively small σ and comparatively large ν are frequently used. It works well to compare a $P_{\lambda,\nu}$ of a numerical value, say, .93 with a corresponding value $\Pi_{\lambda,\nu}$ of, say, .95 In classroom discussions the error may seem insignificant. However, in most actual applications one would consider the complementary probabilities, and the very same figures mean an approximation .05 to the correct value 07. If a confidence limit is set to the five per cent level, the normal approximation would in our example mean that two out of seven critical cases are missed. Consider next the *example* $p = \frac{1}{10}$, $n = 10{,}000$. For values of k around 1120 the relative error of N_k is about .30; it increases rapidly with increasing k. Around $k = 1150$ the relative error exceeds 2/3, around 1180 it is nearly 1.4. And yet this example is conservative in comparison with many cases where the normal approximation is used in practice.

It is surprising that the classical norming (2) is generally accepted although there does not seem to exist any deeper reason for it. The use of moments, though usually very convenient, does not necessarily lead to best results. For example, the density function

$$f_n(x) = \frac{1}{n!} x^n e^{-x} \tag{5}$$

is the $(n + 1)$-fold convolution of $f_0(x)$ with itself and therefore, for large n, of nearly normal "type." The conventional norming would approximate $f_n(x)$ by $\{2\pi(n + 1)\}^{-1/2} e^{-\{x-(n+1)\}^2/2(n+1)}$, while the use of the norming factor n instead of $(n + 1)$ seems clearly indicated

Actually, as will be seen, it is natural (at least for small values of $k - np$) to replace (2) by

$$x_k = \{k + \tfrac{1}{2} - (n + 1)p\}\sigma^{-1}, \tag{6}$$

and accordingly to approximate $P_{\lambda,\nu}$ by the error integral taken between the limits

$$\{\lambda - (n + 1)p\}\sigma^{-1} \quad \text{and} \quad \{\nu + 1 - (n + 1)p\}\sigma^{-1}. \tag{7}$$

For example, let $p = \frac{1}{10}$, $n = 500$, $\lambda = 50$, $\nu = 55$. The correct value is $P_{50,55} \approx .317573$; the norming (2) leads to $\Pi_{50,55} \approx .32357$, while the more natural limits (6) lead to an approximation 31989. More important are the quite unexpected simplifications which the norming (6) permits when one studies the error for large x_k or small σ.

We are now led to reformulate the problem: *instead of starting with arbitrary limits for the error integral and to estimate the resulting error, we shall try to determine the limits so as to minimize the error.* Theoretically, for any given λ, ν these limits could be determined so as to give an exact value for $P_{\lambda,\nu}$. However, such limits would depend in the most intricate way on λ and ν. For practical purposes one would restrict the considerations to certain simple functions such as polynomials.

We shall here consider only the case where the limits are at most quadratic polynomials. Essentially our problem seems that treated by Serge Bernstein (and, apparently, only by him). In a series of papers since 1924, S. Bernstein has considered the accuracy of the normal approximation Quite recently[3] he has, by a considerable computational effort, extended the range of validity from $npq \geq 365$ to $npq \geq 62\ 5$ and proved the following

THEOREM (*S. Bernstein*): *Let*

$$npq \geq 62\ 5 \tag{8}$$

and let α_x, β_x *be the solutions of the quadratic equations*

$$\begin{aligned} x - \tfrac{3}{2} - np &= \alpha_x(npq)^{1/2} + \frac{q-p}{6}\alpha_x^2 \\ x + \tfrac{1}{2} - np &= \beta_x(npq)^{1/2} + \frac{q-p}{6}\beta_x^2 . \end{aligned} \tag{9}$$

If

$$\alpha \geq 0, \qquad \beta < 2^{1/2}(npq)^{1/6} \tag{10}$$

then

$$\Phi(\beta_\nu) - \Phi(\beta_\lambda) \leqq P_{\lambda,\nu} \leq \Phi(\alpha_\nu) - \Phi(\alpha_\lambda). \tag{11}$$

The conditions (10) are practically equivalent to

$$\lambda \geq np + \tfrac{3}{2}, \qquad \nu \leq np + 2^{1/2}\sigma^{1/3} \tag{12}$$

The remarkable feature of this excellent result is that the error remains $O(\sigma^{-1})$ throughout an interval which increases with σ (instead of the conventional uniformly bounded intervals).

In the sequel it will be shown that startling simplifications can be obtained if the norming (6) is used from the beginning instead of (2) Our main result is an improvement of S. Bernstein's theorem The condition (8) will be replaced by $(n + 1)pq \geq 9$ The first condition in (10) will be relaxed to $k \geq (n + 1)p$, that is to say, our theorem will hold for all k exceeding the central value (for those less than the central value an analogous theorem holds); in the other condition (10), the numerical value $2^{1/2}$ will be replaced by an arbitrary constant. Instead of quadratic equations, we shall consider quadratic polynomials. And finally, the gap between the two sets of limits will be reduced.

It will be seen that the computations leading to this improvement are almost negligible in comparison with S Bernstein's deeper method, with slightly more sophisticated arguments and numerical evaluations, our results can be considerably improved. Our consideration will be based on a new expression for T_k, in which only exponential terms appear but the usual square root is missing.

[3] S. Bernstein [1], the first paper of the series appears to have appeared in Učenye Zapiski, Kiev, 1924

In passing from approximations to T_λ to approximations to $P_{\lambda,\nu}$ one has to replace sums by integrals. This procedure is cumbersome if an estimate of the relative error is desired. Euler's formula and other standard formulas are of little use We shall therefore start with a lemma which, it is hoped, may be useful in this connection; it will therefore be proved in a slightly more general form than actually required for the present paper.

2. LEMMA[4] 1. *For* $0 < h < \frac{1}{3}$ *and* $|xh| \leq 1$

$$\int_{x-h/2}^{x+h/2} e^{-u^2/2}\,du = he^{-x^2/2+(x^2-1)h^2/24+\omega h^4}, \tag{13}$$

with

$$-\frac{x^4}{880} \leq \omega \leq \frac{1}{285}. \tag{14}$$

PROOF. Denote the integral in (13) by J. Then

$$h^{-1}e^{x^2/2}J = h^{-1}\int_{-h/2}^{h/2} e^{-xt-t^2/2}\,dt = 2h^{-1}\int_0^{h/2} chxte^{-t^2/2}\,dt. \tag{15}$$

We begin by showing that for $0 \leq \alpha \leq \frac{1}{2}$

$$e^{\alpha^2/2-\alpha^4/11} \leq ch\alpha \leq e^{\alpha^2/2-\alpha^4/16}. \tag{16}$$

In fact

$$e^{\alpha^4/11}ch\alpha \geq \left(1+\frac{\alpha^2}{2}+\frac{\alpha^4}{24}\right)\left(1+\frac{\alpha^4}{11}\right) \geq 1+\frac{\alpha^2}{2}+\frac{\alpha^4}{8}\sum_0^\infty\left(\frac{1}{24}\right)^\rho \geq e^{\alpha^2/2}, \tag{17}$$

and

$$e^{\alpha^2/2-\alpha^4/16} \geq \left(1+\frac{\alpha^2}{2}+\frac{\alpha^4}{8}\right)\left(1-\frac{\alpha^4}{16}\right) \geq 1+\frac{\alpha^2}{2}+\frac{\alpha^4}{24}\cdot\frac{1}{1-\frac{1}{120}} \geq ch\alpha. \tag{18}$$

It follows from (15) and (16) that

$$\begin{aligned} h^{-1}e^{x^2/2}J &\geq 2h^{-1}\int_0^{h/2} e^{(x^2-1)t^2/6-x^4t^4/55}\cdot e^{(x^2-1)t^2/3-4x^4t^4/55}\,dt \\ &\geq 2h^{-1}\int_0^{h/2} e^{(x^2-1)t^2/6-x^4t^4/55}\left\{1+\frac{x^2-1}{3}t^2-\frac{4x^4t^4}{55}\right\}dt \\ &= 2h^{-1}\left[te^{(x^2-1)t^2/6-x^4t^4/55}\right]_0^{h/2} \end{aligned} \tag{19}$$

which proves one part of the lemma.

To obtain an upper estimate we make use of the inequalities

$$e^{(x^2-1)t^2/3} \leq \left(1+\frac{x^2t^2}{3}\right)e^{-t^2/3+x^4t^4/18}$$

[4] The fraction $\frac{1}{3}$ is chosen quite arbitrarily; if h be restricted to $0 < h < 1$ the first member of (14) remains unchanged, while the fraction $\frac{1}{285}$ on the right side has to be replaced by $\frac{1}{264}$.

$$\leq \left(1 + \frac{x^2 t^2}{3}\right)\left(1 - \frac{t^2}{3}\right) e^{x^4 t^4/18} \frac{1 - \frac{t^2}{3} + \frac{t^4}{18}}{1 - \frac{t^2}{3}} \tag{20}$$

$$\leq \left(1 + \frac{x^2 - 1}{3} t^2\right) e^{x^4 t^4/18 + h^4/286}$$

Using (16) and (20), the proof of the second part of the lemma follows from a computation analogous to (19).

For our purposes it is convenient to use Stirling's formula in a form which is not quite the usual one.

LEMMA 2. (*Stirling's formulas*). For $n \geq 4$,

$$n! = (2\pi)^{\frac{1}{2}}(n + \tfrac{1}{2})^{n+\frac{1}{2}} e^{-(n+\frac{1}{2}) - 1/24(n+\frac{1}{2}) + (7/2880)(1+\vartheta_1)/(n+\frac{1}{2})^3}, \tag{21}$$

or

$$n! = (2\pi)^{\frac{1}{2}} n^{n+\frac{1}{2}} e^{-n+1/12n-(1+\vartheta_2)/360n^3} \tag{22}$$

where

$$|\vartheta_i| < \tfrac{1}{6}, \qquad \vartheta \to 0 \quad \text{as} \quad n \to \infty. \tag{23}$$

Formula (21) can be derived from the gamma function or in any other way that leads to the standard form (22).[5]

3. From now on we shall put

$$\sigma^2 = (n + 1)pq \tag{24}$$

$$x_k = \{k + \tfrac{1}{2} - (n + 1)p\}\sigma^{-1}; \tag{25}$$

the subscript k will be omitted whenever no confusion is to be feared. To transform T_k we shall use (21) for the factorials in the denominator, but (22) for $(n + 1)!$ in the numerator.

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[5] A simple proof runs as follows. Put $B_n = n!(n + \frac{1}{2})^{-(n+\frac{1}{2})} e^{n+\frac{1}{2}+1/24(n+\frac{1}{2})}$. Then

$$\log \frac{B_{\rho-1}}{B_\rho} - \sum_{\nu=2}^{\infty} \left\{\frac{1}{6} - \frac{1}{2\nu(2\nu + 1)}\right\} \frac{1}{(2\rho)^{2\nu}} = \frac{7}{60} \frac{1 + \delta_1}{(2\rho)^4}$$

with $0 < \delta_1 < \frac{1}{70}$ if $\rho \geq 5$. From here (21) follows using the fact that

$$\sum_{\rho=n+1}^{\infty} \log \frac{B_{\rho-1}}{B_\rho} = \log B_n - \tfrac{1}{2} \log (2\pi)$$

and that for $n \geq 4$

$$\frac{1 - \delta}{3(n + \frac{1}{2})^3} < \sum_{n+1}^{\infty} \frac{1}{\rho^4} < \frac{1}{3(n + \frac{1}{2})^3}$$

with $0 < \delta < \frac{3}{25}$. In this way the estimate (23) can be considerably improved.

Then

$$\log((2\pi)^{\frac{1}{2}}\sigma T_k) = (n+1)\log(n+1) - (k+\tfrac{1}{2})\log\frac{k+\frac{1}{2}}{p}$$

$$- (n-k+\tfrac{1}{2})\log\frac{n-k+\frac{1}{2}}{q} + \frac{1}{12(n+1)} + \frac{1}{24(k+\frac{1}{2})}$$

$$(26) \qquad + \frac{1}{24(n-k+\frac{1}{2})} - \rho$$

$$= -\frac{\sigma^2}{q}\left(1+\frac{qx}{\sigma}\right)\log\left(1+\frac{qx}{\sigma}\right) - \frac{\sigma^2}{p}\left(1-\frac{px}{\sigma}\right)\log\left(1-\frac{px}{\sigma}\right)$$

$$+ \frac{pq}{12\sigma^2} + \frac{q}{24\sigma^2\left(1+\frac{qx}{\sigma}\right)} + \frac{p}{24\sigma^2\left(1-\frac{px}{\sigma}\right)} - \rho$$

$$(27) \qquad 0 \le \rho \le \frac{7}{6}\left\{\frac{1}{360(n+1)^3} + \frac{7}{2880}\left[\frac{1}{(k+\frac{1}{2})^3} + \frac{1}{(n-k+\frac{1}{2})^3}\right]\right\}$$

$$\le \frac{7}{6}\cdot\frac{1}{360\sigma^6}\left\{p^3q^3 + \frac{7}{8}(p^3+q^3)\right\},$$

provided only that $k \ge 4$, $(n-k) \ge 4$. Asymptotically ρ is equivalent to the right-hand member without factor $\frac{7}{6}$ (which, by the way, could be replaced by $1 + \frac{1}{40}$). Obviously

$$(28) \qquad 0 < \rho < \frac{1}{300\sigma^6},$$

if $k \ge 4$, $n-k \ge 4$. We shall consider later on the case $\sigma \ge 3$, $|x| \le \frac{2}{3}\sigma$; then clearly $k \ge 4$, $n-k \ge 4$, so that the use of (28) will be justified. Expanding (26) into a power series we obtain

THEOREM. *If $k \ge 4$, $n-k \ge 4$,*

$$(29) \qquad T_k = (2\pi)^{-\frac{1}{2}}\sigma^{-1}\exp\left\{-\sum_2^\infty \frac{p^{\nu-1}-(-q)^{\nu-1}}{\nu(\nu-1)}\frac{x^\nu}{\sigma^{\nu-2}} + \frac{1}{24\sigma^2}\sum_3^\infty\{p^{\nu-1}-(-q)^{\nu-1}\}\left(\frac{x}{\sigma}\right)^{\nu-2} + \frac{1+2pq}{24\sigma^2} - \rho\right\}$$

where ρ satisfies (28) *(and* (27)*); x and σ are defined by* (25) *and* (24), *respectively*

Each term of the second series will usually be small as compared to the corresponding term of the first series; the second series can therefore, if desired, be absorbed in the error term. If x is small the first term of the first series will be preponderant. However, as x increases, more and more terms will make themselves noticeable; if $x \sim \sigma^{1/2}$, three terms will be essential, and so on.

Formula (29) permits us to approximate $P_{\lambda,\nu}$ by means of integrals. The tangent rule would suggest to compare $P_{\lambda,\nu}$ to

$$(30) \qquad \Phi\left(x_\nu + \frac{1}{2\sigma}\right) - \Phi\left(x_\mu - \frac{1}{2\sigma}\right),$$

and (29) together with lemma (1) permits easily to estimate the *relative error* in the practically most important cases It is also seen that the limits in (30) are essentially the only limits depending linearly on λ and ν which will render the relative error $O(\sigma^{-1})$ for $a = O(1)$. Instead of elaborating on these simple questions we proceed to the more intricate problem of limits which are quadratic polynomials in λ and ν.

4. For brevity we shall from now on put

$$\frac{p-q}{6} = a \tag{31}$$

The estimate $|a| \leqq \frac{1}{6}$ will be used constantly It obviously suffices to consider values of $\lambda < \nu$ which exceed the central value $[(n+1)p]$.

THEOREM *Suppose that*

$$\sigma > 3 \tag{32}$$

and

$$\lambda \geq (n+1)p \qquad \nu + \tfrac{1}{2} \leq (n+1)p + \tfrac{2}{3}\sigma^2. \tag{33}$$

Then

$$P_{\lambda,\nu} \leq e^{5(1-pq)/36\sigma^2}\{\Phi(\eta_{\nu+1}) - \Phi(\eta_\lambda)\}, \tag{34}$$

if

$$\eta_k = \frac{k-(n+1)p}{\sigma} + \frac{a}{\sigma}\left\{\frac{k-(n+1)p}{\sigma}\right\}^2 + \frac{2a}{\sigma} - \frac{1}{2\sigma^2}, \tag{35}$$

while the inequality in (34) *is reversed if*

$$\eta_k = \frac{k-(n+1)p}{\sigma} + \frac{a}{\sigma}\left\{\frac{k-(n+1)p}{\sigma}\right\}^2 + \frac{2a}{\sigma} + \frac{M}{6\sigma} + \frac{1}{7\sigma}, \tag{36}$$

where

$$M = \frac{x_\nu^3}{\sigma} = \frac{\{\nu + \frac{1}{2} - (n+1)p\}^3}{\sigma^4} \tag{37}$$

The gap between the limits (35) and (36) is $O(\sigma^{-1})$ if $x_\nu^3 = O(\sigma)$. In S. Bernstein's case (12), $M \leq \sqrt{2}$ and the gap is about $2/(5\sigma)$. It will be seen from the proof that it requires only routine computations to improve the correction term $\left\{\frac{M}{6} + \frac{1}{7}\right\}\sigma^{-1}$ in (36)

PROOF. Put

$$\xi_k = x_k + \frac{a}{\sigma}x_k^2, \tag{38}$$

again suppressing the subscripts wherever convenient. As a consequence of (33), we shall be concerned only with values x_k satisfying

$$\frac{1}{2\sigma} < x < \frac{2}{3}\sigma. \tag{39}$$

Consider first the main series in (29) and write

$$\sum_{2}^{\infty} \frac{p^{\nu-1} - (-q)^{\nu-1}}{\nu(\nu-1)} \frac{x^{\nu}}{\sigma^{\nu-2}} = \tfrac{1}{2}\xi^2 + A, \tag{40}$$

where

$$A = \left(\frac{p^3 + q^3}{12} - \frac{a^2}{2}\right)\frac{x^4}{\sigma^4} + \sum_{5}^{\infty} \frac{p^{\nu-1} - (-q)^{\nu-1}}{\nu(\nu-1)} \frac{x^{\nu}}{\sigma^{\nu-2}}. \tag{41}$$

We shall require some estimates of A. First consider the case $a > 0$. Then all terms of the series are positive, while the expression within parentheses assumes its minimum $\frac{1}{48}$ for $p = \frac{1}{2}$. By (39) $\xi < \frac{10}{9} x$, whence

$$A > \frac{1}{74} \frac{\xi^4}{\sigma^2} \quad \text{if} \quad a > 0. \tag{42}$$

If $a < 0$ the signs in the series (41) alternate, each negative term being smaller in absolute value than the preceding positive term. Therefore, using (39),

$$A \geq \left\{\frac{p^3 + q^3}{12} - \frac{a^2}{2} - \frac{q^4 - p^4}{30}\right\} \frac{x^4}{\sigma^2}. \tag{43}$$

The expression within braces is a cubic in p which assumes its minimum for $p = (1 + \sqrt{793})/72 = .405\ldots$ It follows that

$$A \geq \frac{1}{60} \frac{x^4}{\sigma^4} \geq \frac{1}{60} \frac{\xi^4}{\sigma^4} \quad \text{if} \quad a < 0 \tag{44}$$

(half of this estimate would actually suffice for our purposes). On the other hand, it is evident from (41) that the ratio A/x^4 attains its maximum for $p = 1$. Therefore, using (39)

$$A < \frac{2}{15} \frac{x^4}{\sigma^2}. \tag{45}$$

Next we write

$$\frac{1}{24\sigma^2} \sum_{3}^{\infty} \{p^{\nu-1} - (-q)^{\nu-1}\} \left(\frac{x}{\sigma}\right)^{\nu-2} = \frac{a}{4\sigma^3}\xi + B, \tag{46}$$

whence

$$B = \tfrac{1}{2}\left[\frac{p^3 + q^3}{12} - \frac{a^2}{2}\right]\frac{x^2}{\sigma^4} + \frac{1}{24\sigma^2} \sum_{5}^{\infty} \{p^{\nu-1} - (-q)^{\nu-1}\} \left(\frac{x}{\sigma}\right)^{\nu-2}. \tag{47}$$

A trivial computation analogous to (43) shows that $B > 0$. Again, if $a < 0$, the signs in the series (47) alternate and in this case

$$0 \leq B \leq \tfrac{1}{2}\left[\frac{p^3 + q^3}{12} - \frac{a^2}{2}\right]\frac{x^2}{\sigma^4} \leq \frac{5}{144} \frac{x^2}{\sigma^4} < \frac{1}{20} \frac{\xi^2}{\sigma^4}. \tag{48}$$

If $a > 0$ we can majorize (47) by a geometric series and obtain

$$0 \le B \le \frac{1}{8}\frac{x^2}{\sigma^4} \le \frac{1}{8}\frac{\xi^2}{\sigma^4}. \tag{49}$$

Now put

$$\Delta\xi_k = \sigma^{-1}\left(1 + \frac{2a}{\sigma}x_k\right). \tag{50}$$

Then

$$\xi_k + \tfrac{1}{2}\Delta\xi_k = \xi_{k+1} - \tfrac{1}{2}\Delta\xi_{k+1} \tag{51}$$

so that the intervals with endpoints $\xi_k \pm \frac{1}{2}\Delta\xi_k$ are non-overlapping and contiguous. Clearly

$$\Delta\xi = \sigma^{-1}\left\{1 + \frac{4a}{\sigma}\xi\right\}^{1/2}. \tag{52}$$

Introducing (40), (46), and (52) into (29) we obtain

$$\begin{aligned} T_k = (2\pi)^{-1/2}\Delta\xi \exp\left\{-\frac{\xi^2}{\sigma} - A + B + \frac{a}{4\sigma^3}\xi - \tfrac{1}{2}\log\left(1 + \frac{4a\xi}{\sigma}\right)\right. \\ \left. + \frac{1 + 2pq}{24\sigma^2} - \rho\right\}. \end{aligned} \tag{53}$$

To appraise the logarithmic term we write

$$\tfrac{1}{2}\log\left(1 + \frac{4a\xi}{\sigma}\right) = \frac{2a\xi}{\sigma} - C \tag{54}$$

$C\,\xi^{-2}$ attains its maximum value when $a = -\frac{1}{6}$, and it is readily seen that

$$\begin{aligned} 0 < C < \frac{4a^2\xi^2}{\sigma^2} \quad \text{if} \quad a > 0 \\ 0 < C < \frac{6a^2\xi^2}{\sigma^2} \quad \text{if} \quad a < 0. \end{aligned} \tag{55}$$

Finally we put, with a parameter u to be determined,

$$y = \xi + \frac{2a - u}{\sigma}, \qquad \Delta y = \Delta\xi. \tag{56}$$

If one puts

$$u = \frac{1}{2\sigma} - \frac{a}{4\sigma^2} \tag{57}$$

and η_k is defined by (35), then

$$y_k + \tfrac{1}{2}\Delta y_k = \eta_{k+1}, \qquad y_k - \tfrac{1}{2}\Delta y_k = \eta_k. \tag{58}$$

On the other hand, if

$$u = -\frac{M}{6} - \frac{1}{7} - \frac{a}{4\sigma^2} \tag{59}$$

and η_k is defined by (36), the identities (58) hold again. Accordingly, all we have to show is that, with u defined by (57),

$$T_k \leq \{\Phi(y_k + \tfrac{1}{2}\Delta y_k) - \Phi(y_k - \tfrac{1}{2}\Delta y_k)\}e^{5(1-pq)/36\sigma^2} \tag{60}$$

and that the inequality in (60) is reversed if u is defined by (59).

Elementary transformations lead from (53) to

$$T_k = (2\pi)^{-\frac{1}{2}}\Delta y\cdot\exp\left\{-\frac{y^2}{2} + \frac{(\Delta y)^2}{24}(y^2 - 1) + \frac{5(1 - pq)}{36\sigma^2} + E\right\}, \tag{61}$$

where

$$E = \frac{u^2 - 4au}{2\sigma^2} - \left(\frac{u}{\sigma} - \frac{5a}{12\sigma^3}\right)\xi - \frac{1}{24\sigma^2}\left(1 + \frac{4a\xi}{\sigma}\right)y^2 - A + B + C - \rho. \tag{62}$$

Let now u be defined by (57). In view of lemma 1 and (61), the inequality (60) will be proved if we show that

$$E_1 \equiv E + \frac{y^4(\Delta y)^4}{880} \leq 0. \tag{63}$$

Now clearly

$$\frac{y^2}{24\sigma^2}\left(1 + \frac{4a\xi}{\sigma}\right) = \frac{y^2(\Delta y)^2}{24} \geq \frac{y^4(\Delta y)^4}{880}. \tag{64}$$

Moreover, introducing the estimates (28), (32), (42), (44), (48), (49), and (55) into (62) it is seen that for $a > 0$

$$\sigma^2 E_1 < \frac{1}{24\sigma} - \frac{25}{54}\xi + \frac{1}{8}\xi^2 - \frac{1}{74}\xi^4, \tag{65}$$

and for $a < 0$

$$\sigma^2 E_1 < \frac{2}{9\sigma} - \tfrac{1}{2}\xi + \tfrac{1}{5}\xi^2 - \frac{1}{60}\xi^4. \tag{66}$$

The derivatives of the right-hand members in (65) and (66) are both negative for $\xi > 0$ Now we are interested only in values x satisfying (39). For such values $\xi \geq \frac{107}{216\sigma}$. For $\xi = \frac{107}{216\sigma}$ the right-hand members in (65) and (66) are negative, so that $E_1 < 0$ for $x > \frac{1}{2\sigma}$. This proves the first part of our theorem.

The proof that with (59) the inequality in (60) is reversed proceeds on similar lines. We have to show that

$$E_2 \equiv E - \frac{(\Delta y)^4}{285} \geq 0. \tag{67}$$

Suppose that $a < 0$, which is the less favorable case. Then, by (45), (37), and (39),

$$A \leq \frac{2M}{15}\frac{x}{\sigma} \leq \frac{3M}{20}\cdot\frac{\xi}{\sigma}. \tag{68}$$

Similarly

$$\frac{1}{24\sigma^2}\left(1 + \frac{4a\xi}{\sigma}\right)y^2 \leq \frac{1}{24\sigma^2}\left(\xi - \frac{u}{\sigma}\right)^2 \tag{69}$$

Using (62) we have therefore, neglecting the non-negative terms B and C,

$$\begin{aligned} E_2 \geq {} & \frac{u^2}{2\sigma^2} - \frac{u^2}{24\sigma^4} - \frac{u}{3\sigma^2} - \frac{1}{250\sigma^4} \\ & + \xi\left\{-\frac{u}{\sigma} - \frac{5}{72\sigma^3} - \frac{3M}{20\sigma} + \frac{u}{12\sigma^3}\right\} - \frac{1}{24\sigma^2}\xi^2. \end{aligned} \tag{70}$$

The expression at the right side represents a parabola, and it suffices to show that it assumes positive values at the endpoints of our interval (39). Now

$$\frac{u^2}{2}\left(1 - \frac{1}{12\sigma^2}\right) - \frac{u}{3} \geq -\frac{1}{18}\frac{1}{1 - \frac{1}{12\sigma^2}} > -\frac{6}{107}, \tag{71}$$

and simple arithmetic shows that, with (59) the expression within the braces more than counterbalances the negative terms outside.[6] If $a > 0$ the situation is more favorable and the estimate (59) can then be further improved.

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[6] A more careful computation shows that it suffices if we put $u = -\frac{M}{6} - \frac{1}{8} - \frac{a}{4\sigma^2}$ instead of $-\frac{M}{6} - \frac{1}{7} - \frac{a}{4\sigma^2}$.

THE VARIANCE OF THE MEASURE OF A TWO-DIMENSIONAL RANDOM SET

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1. Introduction. In a recent paper H. E. Robbins[1] has solved the problem of the variance of the measure of a one-dimensional random set. The present paper treats a similar problem relating to a two-dimensional random set under somewhat more general conditions.

Let R denote a rectangle of dimensions $a \times b$ whose position is fixed. Let R' denote another fixed rectangle concentric with R, its sides $a + \gamma$ and $b + \gamma$ (where $\gamma > 0$) being parallel to the sides a and b respectively of R. Finally, let ρ denote a rectangle of fixed dimensions but variable position, whose sides $\alpha < 2\gamma$ and $\beta < 2\gamma$ are parallel to a and b respectively, but the position of whose center will be considered as random. In fact it will be assumed that the rectangle ρ is dropped on the plane of R in a manner which satisfies the following two assumptions:

(i) The probability that the center of ρ falls within R' exactly s times has a defined value P_s for each $s = 0, 1, 2, \cdots$ Thus, if $\Psi(u)$ denotes the probability generating function of s, so that

$$\Psi(u) = \sum_{s=0}^{\infty} u^s P_s, \tag{1}$$

then $\Psi(u)$ is assumed known but will be left arbitrary till the general result is obtained.

(ii) Whenever a fixed number s of centers of ρ fall within R', it will be assumed that the probability that exactly k centers of ρ fall within any chosen sub-area w contained in R' is given by the binomial expression

$$\frac{s!}{k!(s-k)!}\frac{w^k}{R'}\left(1 - \frac{w}{R'}\right)^{s-k} \tag{2}$$

Under the above conditions, denote by E the set of all those points of R which are covered at least once by the rectangle ρ during the course of the trials considered. Let X denote the measure of E. The purpose of this paper is to evaluate the first two moments of X.

First, the computations will be made for the case when s is fixed, i.e. when

$$\Psi(u) = u^s. \tag{3}$$

The values of the two moments of X computed for fixed s will be denoted by $M_1(a,b|s)$ and $M_2(a,b|s)$. Next, the moments of X will be evaluated for an arbitrary generating function $\Psi(u)$, and these will be denoted by $M_1(a, b)$ and $M_2(a, b)$.

[1] H. E. Robbins, "On the measure of a random set", *Annals of Math. Stat.*, Vol. 15 (1944), pp. 70–74.

H E. Robbins has found the first moment

$$M_1(a, b \mid s) = ab\left\{1 - \left(1 - \frac{\alpha\beta}{R'}\right)^s\right\} \tag{4}$$

Also, for a one-dimensional set, he has obtained the second moment, say $M_2(a|s)$, when $\alpha \leq a$.

It follows immediately from (4) and (1) that, whatever be the probability generating function $\Psi(u)$,

$$M_1(a, b) = ab\left\{1 - \Psi\left(1 - \frac{\alpha\beta}{R'}\right)\right\} \tag{5}$$

In particular, if the probabilities P_s are those of Poisson when the density of positions of the center of ρ per unit of area is λ, so that

$$\Psi(u) = e^{\lambda R'(u-1)}, \tag{6}$$

then

$$M_1(a,b) = ab\{1 - e^{-\alpha\beta\lambda}\} \tag{7}$$

Our remaining problem, therefore, is that of evaluating the second moment of X. Instead we shall evaluate the second moment of

$$Y = ab - X, \tag{8}$$

and shall denote it by $m(a, b \mid s)$ or $m(a, b)$ according as s is or is not considered to be fixed.

2. Derivative of the second moment of Y. In order to evaluate $m(a, b)$, we begin by calculating its second (mixed) derivative, say $D(a, b \mid s)$, where

$$\begin{aligned} D(a, b \mid s) &= \frac{\partial^2 m(a, b \mid s)}{\partial a \partial b} \\ &= \lim_{\Delta a, \Delta b \to 0} \frac{1}{\Delta a \Delta b} \{m(a + \Delta a, b + \Delta b \mid s) - m(a, b + \Delta b \mid s) \\ &\qquad - m(a + \Delta a, b \mid s) + m(a, b \mid s)\} \\ &= \lim \frac{1}{\Delta a \Delta b} I(\Delta a, \Delta b) \quad \text{(say)}, \end{aligned} \tag{9}$$

where Δa and Δb are the increments of a and b respectively. Once $D(a, b \mid s)$ is found, the formula for $m(a, b \mid s)$ will be obtained by two quadratures. For definiteness we shall assume Δa and Δb both to be positive, but of course the argument which follows applies equally to other cases.

Consider the rectangle of dimensions $(a + \Delta a)$ and $(b + \Delta b)$ as shown in Figure 1, and denote by U, V and W the measures of the "uncovered" parts of the three rectangles $\Delta a \times b$, $a \times \Delta b$, and $\Delta a \times \Delta b$ respectively. That is to say, U, V and W are defined with respect to these three rectangles precisely in the same

manner in which Y is defined with respect to the original rectangle $a \times b \equiv R$. Using the letter E to denote the expectation, we easily find that

$$I(\Delta a, \Delta b) = 2E(YW) + 2E(UV) + 2E(VW) + 2E(UW) + E(W^2). \tag{10}$$

However, each of the three expectations in the second line of formula (10) is infinitesimal of an order higher than the product $\Delta a \Delta b$. In fact, none of the

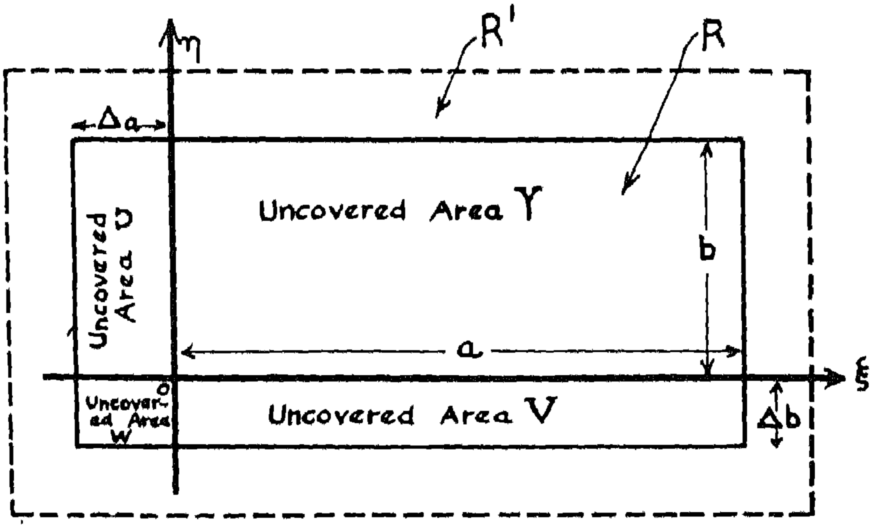

FIGURE 1.

variables U, V and W can exceed the area of the rectangle of which it forms part, that is,

$$\begin{aligned} 0 &\leq U \leq b\Delta a, \\ 0 &\leq V \leq a\Delta b, \\ 0 &\leq W \leq \Delta a \Delta b. \end{aligned} \tag{11}$$

It follows that

$$\begin{aligned} 0 &\leq E(UW) \leq b(\Delta a)^2 \Delta b, \\ 0 &\leq E(VW) \leq a\Delta a(\Delta b)^2, \\ 0 &\leq E(W^2) \leq (\Delta a \Delta b)^2. \end{aligned} \tag{12}$$

Hence, from (9), (10) and (12)

$$D(a, b \mid s) = 2 \lim \frac{1}{\Delta a \Delta b} \{E(YW) + E(UV)\}. \tag{13}$$

We now reduce the calculation of (13) to finite form by approximating to the infinite sets Y, U, V, W by progressively more ample but finite sets. To do so,

we cover R' by progressively more ample but finite networks of points. More precisely: consider a rectangular system of axes $O\xi$ and $O\eta$ oriented as in Figure 1 so that the axes are common boundaries of $a \times b \equiv R$ and of the rectangles obtained by increasing a and b Let

$$d_n = a/(n+1), \qquad \delta_n = b/(n+1). \tag{14}$$

Consider the lattice of points (ij) with coordinates

$$\xi_i^{(n)} = id_n, \qquad \eta_j^{(n)} = j\delta_n \tag{15}$$

for $i = -v_1^{(n)}, -v_1^{(n)} + 1, \cdots, 0, 1, 2, \cdots, n; j = -v_2^{(n)}, -v_2^{(n)} + 1, \cdots, 0, 1, 2, \cdots, n$, where $v_1^{(n)}$ and $v_2^{(n)}$ are the greatest integers such that

$$v_1^{(n)} d_n \leq \Delta a \tag{16}$$

and

$$v_2^{(n)} \delta_n \leq \Delta b \tag{17}$$

To simplify the writing, the superscripts (n) will henceforth be dropped.

With every point (ij) we associate a random variable x_{ij} defined as follows If in the course of the trials contemplated none of the rectangles ρ covers (ij), then $x_{ij} = 1$. Otherwise $x_{ij} = 0$. Further, write

$$\begin{aligned} Y_n &= d_n \delta_n \sum_{i=0}^{n} \sum_{j=0}^{n} x_{ij}, \\ U_n &= d_n \delta_n \sum_{i=-v_1}^{0} \sum_{j=0}^{n} x_{ij}, \\ V_n &= d_n \delta_n \sum_{i=0}^{n} \sum_{j=-v_2}^{0} x_{ij}, \\ W_n &= d_n \delta_n \sum_{i=-v_1}^{0} \sum_{j=-v_2}^{0} x_{ij}. \end{aligned} \tag{18}$$

Now the boundary of the set E, for a fixed s, consists of one or more polygons having a finite total number of sides each of bounded length. It follows that, given any $\epsilon > 0$, there exists, for a fixed s, a number $N_\epsilon(s)$ such that $n > N_\epsilon(s)$ implies that

$$| Y_n - Y | < \epsilon, \tag{19}$$

with similar inequalities relating to U_n, V_n and W_n. Hence it follows immediately that

$$\begin{aligned} \lim_{n \to \infty} E(Y_n W_n \mid s) &= E(YW \mid s), \\ \lim_{n \to \infty} E(U_n V_n \mid s) &= E(UV \mid s). \end{aligned} \tag{20}$$

The expectations in formula (13) will therefore be obtained as limits of those on the left hand sides of (20). We have

$$E(Y_n W_n \mid s) = d_n^2 \delta_n^2 \sum_{i=-v_1}^{0} \sum_{j=-v_2}^{0} E\left(x_{ij} \sum_{k=0}^{n} \sum_{l=0}^{n} x_{kl} \mid s\right), \tag{21}$$

$$E(U_n V_n \mid s) = d_n^2 \delta_n^2 \sum_{i=-v_1}^{0} \sum_{j=0}^{n} E\left(x_{ij} \sum_{k=0}^{n} \sum_{l=-v_2}^{0} x_{kl} \mid s\right). \tag{22}$$

Hitherto we have made no assumptions concerning the values of Δa and Δb. Since these are to tend to zero, we may assume that

$$\begin{aligned} 0 &< \Delta a < \gamma - \alpha/2, \\ 0 &< \Delta b < \gamma - \beta/2. \end{aligned} \tag{23}$$

On this assumption, we shall now compute the expectations of the type $E(x_{ij}x_{kl} \mid s)$, of which (21) and (22) are linear combinations.

Since the variables x_{ij} and x_{kl} are capable only of the two values unity and zero, the expectation of their product is simply the probability that both of them are equal to unity, i.e the probability that both points (ij) and (kl) are "missed" by all the s rectangles ρ falling on R'. This probability may have one of two forms. If both

$$d_n \mid i - k \mid < \alpha \quad \text{and} \quad \delta_n \mid j - l \mid < \beta, \tag{24}$$

then

$$E(x_{ij}x_{kl} \mid s) = \left\{1 - \frac{2\alpha\beta - (\alpha - d_n \mid i - k \mid)(\beta - \delta_n \mid j - l \mid)}{R'}\right\}^s; \tag{25}$$

while otherwise

$$E(x_{ij}x_{kl} \mid s) = \left(1 - \frac{2\alpha\beta}{R'}\right)^s; \tag{26}$$

in each case, in virtue of the assumption (ii) of Section 1.

The essential content of equations (24) to (26) is that, once the other variables appearing in them are assigned, $E(x_{ij}x_{kl} \mid s)$ is a function only of the differences $i - k$ and $j - l$ It is this fact which allows us to evaluate the limits of the quantities in (21) and (22) in a simple manner, in effect by holding one of the two freely variable points (ij), (kl) in a fixed position, say at the origin. Thus, let

$$E(\theta_n \mid s) = d_n^2 \delta_n^2 \sum_{i=-v_1}^{0} \sum_{j=-v_2}^{0} E\left(x_{ij} \sum_{k=i}^{n+i} \sum_{l=j}^{n+j} x_{kl} \mid s\right). \tag{27}$$

Owing to the remark just made, the expectation

$$E\left(x_{ij} \sum_{k=i}^{n+i} \sum_{l=j}^{n+j} x_{kl} \mid s\right) = E\left(x_{00} \sum_{k=0}^{n} \sum_{l=0}^{n} x_{kl} \mid s\right) \tag{28}$$

and it follows that

$$(29)\quad \begin{aligned} E(\theta_n \mid s) &= (v_1 + 1)(v_2 + 1)\, d_n^2 \delta_n^2 E\left(x_{00} \sum_{k=0}^{n} \sum_{l=0}^{n} x_{kl} \mid s\right) \\ &= [(v_1 + 1)(v_2 + 1) d_n \delta_n] \left[d_n \delta_n \sum_{k=0}^{n} \sum_{l=0}^{n} E(x_{00} x_{kl} \mid s) \right]. \end{aligned}$$

Of the two factors in the square brackets in (29), the first tends to $\Delta a \Delta b$ as n tends to infinity, and the second tends to the integral

$$(30)\quad \int_0^a \int_0^b f^s(\xi, \eta)\, d\xi d\eta$$

where

$$(31)\quad f(\xi, \eta) \equiv 1 - \frac{2\alpha\beta - (\alpha - \xi)(\beta - \eta)}{R'}$$

if both $0 \le \xi \le \alpha$ and $0 \le \eta \le \beta$, and

$$(32)\quad f(\xi,\eta) \equiv \left(1 - \frac{2\alpha\beta}{R'}\right)$$

otherwise Thus the computation of the limit of $E(\theta_n \mid s)$ is straightforward It remains to show that it differs from that of $E(Y_n W_n \mid s)$ in equation (21) by an infinitesimal which is of an order higher than the product $\Delta a \Delta b$.

Since the variables x_{ik} are capable only of the two values unity and zero the absolute value of the difference between the brackets in (21) and (27), that is, between

$$(33)\quad x_{ij} \sum_{k=0}^{n} \sum_{l=0}^{n} x_{kl} \text{ and } x_{ij} \sum_{k=i}^{n+i} \sum_{l=j}^{n+j} x_{kl},$$

cannot be greater than $-n(i + j) \le n(v_1 + v_2)$. It follows that

$$(34)\quad \left| E(Y_n W_n \mid s) - E(\theta_n \mid s) \right| \le [d_n \delta_n (v_1 + 1)(v_2 + 1)][n \delta_n v_1 d_n + n d_n v_2 \delta_n].$$

As n tends to infinity, the right hand side of (34) tends to the product

$$(35)\quad \Delta a \Delta b [b \Delta a + a \Delta b];$$

whence

$$(36)\quad \begin{aligned} \lim_{\Delta a, \Delta b \to 0} \frac{1}{\Delta a \Delta b} \{\lim_{n \to \infty} E(\theta_n \mid s)\} &= \lim_{\Delta a, \Delta b \to 0} \frac{1}{\Delta a} \frac{1}{\Delta b} E(YW \mid s) \\ &= \int_0^a \int_0^b f^s(\xi, \eta)\, d\xi d\eta. \end{aligned}$$

A very similar procedure will serve to evaluate the limit of $E(UV \mid s)/\Delta a \Delta b$. Here, we replace the two freely variable points (ij), (kl) by two semi-fixed points,

one being restricted to the axis $O\xi$ and the other to the axis $O\eta$. More precisely, instead of considering $E(U_nV_n \mid s)$ in equation (22) we consider, say,

$$E(\phi_n \mid s) = d_n^2\, \delta_n^2 \sum_{i=-v_1}^{0} \sum_{j=0}^{n} E\Big(x_{ij} \sum_{k=i}^{n+i} \sum_{l=-v_2}^{0} x_{kl}\Big) \tag{37}$$

and it is easy to see that

$$\lim_{n\to\infty} \mid E(U_n V_n \mid s) - E(\phi_n \mid s) \mid \le b(\Delta a)^2 (\Delta b), \tag{38}$$

so that the quantity (37) may be used in equations (13) and (20) in place of the quantity (22). However, since $E(x_{ij}x_{kl} \mid s)$ depends only on the differences $i - k$ and $j - l$,

$$E(x_{ij} \sum_{k=i}^{n+i} \sum_{l=-v_2}^{0} x_{kl}) = E\Big(x_{0j} \sum_{k=0}^{n} \sum_{l=-v_2}^{0} x_{kl}\Big) \tag{39}$$

and therefore

$$E(\phi_n \mid s) = \{d_n(v_1 + 1)\}\Big\{d_n\, \delta_n^2 \sum_{j=0}^{n} E\Big(x_{0j} \sum_{k=0}^{n} \sum_{l=-v_2}^{0} x_{kl} \mid s\Big)\Big\} \tag{40}$$

Further, and in the same way, we may replace the sum in (40), namely

$$\sum_{j=0}^{n} E\Big(x_{0j} \sum_{k=0}^{n} \sum_{l=-v_2}^{0} x_{kl} \mid s\Big) = \sum_{k=0}^{n} \sum_{l=-v_2}^{0} E\Big(x_{kl} \sum_{j=0}^{n} x_{0j} \mid s\Big) \tag{41}$$

by the simpler sum

$$\begin{aligned} \sum_{k=0}^{n} \sum_{l=-v_2}^{0} E\Big(x_{kl} \sum_{j=l}^{n+l} x_{0j} \mid s\Big) &= (v_2 + 1) \sum_{k=0}^{n} E\Big(x_{k0} \sum_{j=0}^{n} x_{0j} \mid s\Big) \\ &= (v_2 + 1 \sum_{k=0}^{n} \sum_{j=0}^{n} E(x_{k0}\, x_{0j} \mid s). \end{aligned} \tag{42}$$

It follows that we may replace the limit of $E(U_nV_n \mid s)$ as expressed in (22) by

$$\lim_{n\to\infty} \{d_n\,(v_1 + 1)\,\delta_n\,(v_2 + 1)\}\Big\{d_n\,\delta_n \sum_{k=0}^{n} \sum_{j=0}^{n} E(x_{k0}\, x_{0j} \mid s)\Big\}, \tag{43}$$

and this is easily found to be equal to

$$\Delta a \Delta b \int_0^a \int_0^b f^{\bullet}(\xi, \eta)\, d\xi d\eta, \tag{44}$$

where $f(\xi, \eta)$ is defined by the formulae (31) and (32)

Collecting this result with that expressed by (36), and substituting in equation (13), we therefore have finally

$$D(a,b \mid s) = 4 \int_0^a \int_0^b f^{\bullet}(\xi, \eta)\, d\xi d\eta. \tag{45}$$

3. The forms of the derivative. Since the function $f(\xi,\eta)$ has two different forms (31) or (32) depending on the relationships between a, b, α and β, it will be necessary to distinguish four different forms of the derivative (45), and of its integral.

First, for values of a and b for which simultaneously

$$a \leq \alpha \quad \text{and} \quad b \leq \beta, \tag{46}$$

the integrand in (45) has the form (31) for the whole region of integration. Hence the value of $D(a, b \mid s)$ in the region (46) is given by, say

$$\begin{aligned} D_1 &= 4\int_0^a \int_0^b \left(1 - \frac{2\alpha\beta - (\alpha - \xi)(\beta - \eta)}{R'}\right)^s d\xi d\eta \\ &= 4\int_{\alpha-a}^{\alpha} \int_{\beta-b}^{\beta} g^s(t, \tau)\, dt d\tau, \end{aligned} \tag{47}$$

where

$$g(t, \tau) \equiv 1 - \frac{2\alpha\beta - t\tau}{R'}. \tag{48}$$

Next, when $a \geqq \alpha$ but $b \leq \beta$, the integrand in (45) has the form determined by (31) only when

$$0 \leq \xi \leq \alpha, \qquad 0 \leq \eta \leq b, \tag{49}$$

whereas when

$$\alpha \leq \xi \leq a, \qquad 0 \leq \eta \leq b, \tag{50}$$

the appropriate form is that determined by (32). Therefore here $D(a, b \mid s)$ has the form, say,

$$D_2 = 4b(a - \alpha)\left(1 - \frac{2\alpha\beta}{R'}\right)^s + 4\int_0^{\alpha} \int_{\beta-b}^{\beta} g^s(t, \tau)\, dt d\tau, \tag{51}$$

Similarly, for

$$a \leq \alpha \quad \text{but} \quad b \geqq \beta, \tag{52}$$

$D(a, b \mid s)$ is given by, say,

$$D_3 = 4a(b - \beta)\left(1 - \frac{2\alpha\beta}{R'}\right)^s + 4\int_{\alpha-a}^{\alpha} \int_0^{\beta} g^s(t, \tau)\, dt d\tau. \tag{53}$$

Finally, in the region in which simultaneously

$$a \geqq \alpha \quad \text{and} \quad b \geqq \beta, \tag{54}$$

$D(a, b \mid s)$ has the form, say,

$$D_4 = 4(ab - \alpha\beta)\left(1 - \frac{2\alpha\beta}{R'}\right)^s + 4\int_0^{\alpha} \int_0^{\beta} g^s(t, \tau)\, dt d\tau. \tag{55}$$

4. The second moment of Y. We have now to determine $m(a, b \mid s)$ for all non-negative values of a and b, from the equation

$$\frac{\partial^2 m(a, b \mid s)}{\partial a \, \partial b} = D(a, b \mid s). \tag{56}$$

The general solution of this equation is

$$m(a, b \mid s) = \int_0^a \int_0^b D(a, b \mid s) \, da db + A(a) + B(b), \tag{57}$$

where $A(a)$ and $B(b)$ are each functions of one variable. These functions are determined by the boundary conditions, namely

$$m(a, 0 \mid s) = m(0, b \mid s) = \frac{\partial m(a, 0 \mid s)}{\partial a} = \frac{\partial m(0, b \mid s)}{\partial b} = 0, \tag{58}$$

which are a consequence of the inequality $0 \leq Y \leq ab$. It is then easily found that the only solution $m(a, b \mid s)$ satisfying (57) and (58) has the following four different forms, depending on the values of a and b.

If $a \leq \alpha$ and $b \leq \beta$, then

$$m(a, b \mid s) = \int_0^a \int_0^b D_1(x, y) \, dxdy = m_1(a, b \mid s) \quad \text{(say)}. \tag{59}$$

If $a \geqq \alpha$ and $b \leq \beta$, then

$$\begin{aligned} m(a, b \mid s) &= m_1(\alpha, b \mid s) + \int_\alpha^a \int_0^b D_2(x, y) \, dxdy \\ &= m_2(a, b \mid s) \quad \text{(say)}. \end{aligned} \tag{60}$$

If $a \leq \alpha$ and $b \geqq \beta$, then

$$\begin{aligned} m(a, b \mid s) &= m_1(a, \beta \mid s) + \int_0^a \int_\beta^b D_3(x, y) \, dxdy \\ &= m_3(a, b \mid s) \quad \text{(say)}. \end{aligned} \tag{61}$$

Finally, if $a \geqq \alpha$ and $b \geqq \beta$, then

$$\begin{aligned} m(a, b \mid s) = m_1(\alpha, \beta \mid s) &+ \int_\alpha^a \int_0^\beta D_2(x, y) \, dxdy + \int_0^\alpha \int_\beta^b D_3(x, y) \, dxdy \\ &+ \int_\alpha^a \int_\beta^b D_4(x, y) \, dxdy = m_4(a, b \mid s) \quad \text{(say)}. \end{aligned} \tag{62}$$

The procedure used to evaluate the integrals (59) to (62) follows the same general pattern, and we shall confine ourselves to outlining it in one case, say (59). There

$$\begin{aligned} m_1(a, b \mid s) &= \int_0^a \int_0^b D_1(x, y) \, dxdy \\ &= 4 \int_0^a \int_0^b dxdy \int_{\alpha - x}^\alpha \int_{\beta - y}^\beta g^s(t, \tau) \, dt \, d\tau \\ &= 4 \int_0^a dx \int_{\alpha - x}^\alpha dt \left\{ \int_0^b dy \int_{\beta - y}^\beta g^s(t, \tau) \, d\tau \right\}. \end{aligned} \tag{63}$$

Integrating the double integral in the braces by parts for y we get, say,

$$I(t) \equiv \int_0^b dy \int_{\beta-y}^{\beta} g^s(t, \tau)\, d\tau = \left[y \int_{\beta-y}^{\beta} g^s(t, \tau)\, d\tau \right]_0^b - \int_0^b y g^s(t, \beta - y)\, dy, \tag{64}$$

whence, substituting $\beta - y = \tau$ in the last integral,

$$\begin{aligned} I(t) &= b \int_{\beta-b}^{\beta} g^s(t, \tau)\, d\tau - \int_{\beta-b}^{\beta} (\beta - \tau) g^s(t, \tau)\, d\tau \\ &= \int_{\beta-b}^{\beta} (\tau + b - \beta) g^s(t, \tau)\, d\tau. \end{aligned} \tag{65}$$

Proceeding now in the same manner with the other double integration in (63), we conclude that

$$\begin{aligned} m_1(a, b \mid s) &= 4 \int_0^a dx \int_{\alpha-x}^{\alpha} I(t)\, dt = 4 \int_{\alpha-a}^{\alpha} (t + a - \alpha) I(t)\, dt \\ &= 4 \int_{\alpha-a}^{\alpha} dt \int_{\beta-b}^{\beta} (t + a - \alpha)(\tau + b - \beta) g^s(t, \tau)\, d\tau, \end{aligned} \tag{66}$$

where, throughout, $g(t, \tau)$ is defined by (48).

Formulae for $m_2(a, b \mid s)$, $m_3(a, b \mid s)$ and $m_4(a, b \mid s)$ are obtained by a similar procedure. They may conveniently be summarized in the following single expression. Define a symbol $[x]$ for any real number x by the equations

$$\begin{aligned} [x] &= x \quad \text{if} \quad x \geqq 0 \\ [x] &= 0 \quad \text{if} \quad x \leqq 0. \end{aligned} \tag{67}$$

With this notation, whatever be the relation between a, b, α and β, we have

$$\begin{aligned} m(a, b \mid s) &= 4 \int_{[\alpha-a]}^{\alpha} \int_{[\beta-b]}^{\beta} (t + a - \alpha)(\tau + b - \beta) \left\{ 1 - \frac{2\alpha\beta - t\tau}{R'} \right\}^s dt d\tau \\ &\quad + \{a^2[b - \beta]^2 + b^2[a - \alpha]^2 - [a - \alpha]^2[b - \beta]^2\} \left(1 - \frac{2\alpha\beta}{R'} \right)^s. \end{aligned} \tag{68}$$

We now allow s to take all values $s = 0, 1, 2, \cdots$ with probabilities P_s given by the generating function (1). Then it follows, from the form of (68), that

$$\begin{aligned} m(a, b) &= 4 \int_{[\alpha-a]}^{\alpha} \int_{[\beta-b]}^{\beta} (t + a - \alpha)(\tau + b - \beta) \Psi \left(1 - \frac{2\alpha\beta - t\tau}{R'} \right) dt d\tau \\ &\quad + \{a^2[b - \beta]^2 + b^2[a - \alpha]^2 - [a - \alpha]^2[b - \beta]^2\} \Psi \left(1 - \frac{2\alpha\beta}{R'} \right). \end{aligned} \tag{69}$$

On subtracting from this the square of the first moment of Y, which by (5) and (8) is

$$ab\Psi\left(1 - \frac{\alpha\beta}{R'}\right),$$

we obtain the variance σ_Y^2 of Y. But the variance of Y is necessarily equal to the variance σ_X^2 of X.

5. Particular cases. (i) $\Psi_1(u) = u^s$. This is the case, considered originally, in which the number s of centers of the rectangles ρ falling within R' is fixed. The explicit evaluation of the variance σ_X^2 depends in this case on the evaluation of the integral

$$\int_{[\alpha-a]}^{\alpha} \int_{[\beta-b]}^{\beta} (t + a - \alpha)(\tau + b - \beta) \left\{ \left(1 - \frac{2\alpha\beta}{R'}\right) + \frac{t\tau}{R'} \right\}^s dt d\tau. \tag{70}$$

The evaluation is easy if one expands the binomial under the sign of the integral and integrates term by term. Each such integral is a product of two simple integrals.

(ii) $\Psi_2(u) = e^{\lambda R'(u-1)}$, *Poisson Case.* This is the case where the probabilities P_s that there are exactly s centers of rectangles ρ within R' are given by the Poisson Law, $P_s = (\lambda R')^s e^{-\lambda R'}/s!$. Substituting the expression of the probability generating function into (69), we obtain for this case

$$\begin{aligned} m(a, b) = {} & 4e^{-2\alpha\beta\lambda} \int_{[\alpha-a]}^{\alpha} \int_{[\beta-b]}^{\beta} (t + a - \alpha)(\tau + b - \beta) \sum_{s=0}^{\infty} \frac{(\lambda t\tau)^s}{s!} dt d\tau \\ & + e^{-2\alpha\beta\lambda}\{a^2[b - \beta]^2 + b^2[a - \alpha]^2 - [a - \alpha]^2[b - \beta]^2\}. \end{aligned} \tag{71}$$

On performing the integration term by term, and contracting the first term of the resulting infinite series into the second line of equation (71), we readily obtain the result

$$\begin{aligned} m(a, b) = {} & 4e^{-2\alpha\beta\lambda} \sum_{s=1}^{\infty} \frac{(\lambda\alpha\beta)^s}{s!} \frac{\alpha\beta}{(s+1)^2(s+2)^2} \\ & \times \left\{ (s + 2)a - \alpha + [\alpha - a]\left(1 - \frac{a}{\alpha}\right)^{s+1} \right\} \\ & \times \left\{ (s + 2)b - \beta + [\beta - b]\left(1 - \frac{b}{\beta}\right)^{s+1} \right\} + e^{-2\gamma\beta\lambda} a^2 b^2, \end{aligned} \tag{72}$$

where $[x]$ continues to have the meaning defined by (67). In virtue of equations (7) and (8), however, the last term of the expression (72) is precisely the square of the first moment of Y when s is Poisson distributed. Hence, for s Poisson distributed, we have the expression for the variance of Y and of X,

$$\begin{aligned} \sigma_Y^2 = \sigma_X^2 = {} & 4e^{-2\alpha\beta\lambda} \sum_{s=1}^{\infty} \frac{(\lambda\alpha\beta)^s}{s!} \frac{\alpha\beta}{(s+1)^2(s+2)^2} \\ & \times \left\{ (s + 2)a - \alpha + [\alpha - a]\left(1 - \frac{a}{\alpha}\right)^{s+1} \right\} \\ & \times \left\{ (s + 2)b - \beta + [\beta - b]\left(1 - \frac{b}{\beta}\right)^{s+1} \right\}. \end{aligned} \tag{73}$$

(iii) $\Psi_3(u) = e^{m(e^{\lambda R'(u-1)}-1)}$, *Contagious case.* This is the case where the probabilities P_s that there are exactly s centers of rectangles ρ within R' are given by the contagious law of type A with two parameters[2]. The evaluation of the second moment of Y is made easy by noticing that the probability generating function appropriate to the contagious distribution may be expressed as a series in terms of the probability generating function of the Poisson Law

$$\begin{aligned}\Psi_3(u) &= e^{-m} \sum_{k\geq 0} \frac{m^k}{k!} \Psi_2^k(u) \\ &= e^{-m} \sum_{k\geq 0} \frac{m^k}{k!} e^{k\lambda R'(u-1)}.\end{aligned} \tag{74}$$

Thus the evaluation of the integral intervening in the formula for the second moment of Y is reduced in the present case to that of formula (71).

6. Remarks on other cases. (i) It may be of interest, in amplification of H. E. Robbins' results, to exhibit the analogues of formulas (68), (69) and (73) in the one-dimensional case. For this case, then, if the interval a is embedded in a larger interval a', we obtain by similar methods beginning with the calculation of $\dfrac{\partial m(a \mid s)}{\partial a}$,

$$m(a \mid s) = 2 \int_{[\alpha-a]}^{\alpha} (t + a - \alpha)\left(1 - \frac{2\alpha - t}{a'}\right)^s dt + [a - \alpha]^2\left(1 - \frac{2\alpha}{a'}\right)^s, \tag{75}$$

whence

$$m(a) = 2 \int_{[\alpha-a]}^{\alpha} (t + a - \alpha)\Psi\left(1 - \frac{2\alpha - t}{a'}\right) dt + [a - \alpha]^2 \Psi\left(1 - \frac{2\alpha}{a'}\right); \tag{76}$$

in particular, if s is Poisson distributed,

$$\begin{aligned}\sigma_{Y}^2 = \sigma_Y^2 = 2e^{-2\alpha\lambda} \sum_{s=1}^{\infty} \frac{(\alpha\lambda)^s}{s!} &\frac{\alpha}{(s+1)(s+2)} \\ &\times \left\{(s+2)a - \alpha + [\alpha - a]\left(1 - \frac{a}{\alpha}\right)^{s+1}\right\}.\end{aligned} \tag{77}$$

The close parallel between these formulas and those for two dimensions make it natural to conjecture analogous formulas for n dimensions, but we have not attempted to establish such formulas.

(ii) For the evaluation of the higher moments of Y it may be useful to notice that precisely the same method as that described above leads to the conclusion that the derivative of the n-th non central moment of Y is

$$\frac{\partial^2 m_n(a, b)}{\partial a \partial b} = \lim_{\Delta a, \Delta b \to 0} \frac{1}{\Delta a \Delta b} \{nE(X^{n-1} W) + n(n-1)E(X^{n-2} UV)\}. \tag{78}$$

[2] J. Neyman, "On a new class of contagious distributions", *Annals of Math Stat*, Vol 10 (1939) pp 35–57

ON THE MEASURE OF A RANDOM SET. II

By H. E. Robbins

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1. Introduction. In a recent paper[1] the author derived general formulas for the moments of the measure of any random set X, and applied the formulas to find the mean and variance of a random sum of intervals on the line. In a subsequent paper[2] J. Bronowski and J. Neyman, using other methods, found the variance when X is a random sum of rectangles in the plane, and raised the question of finding the variance when X is a random sum of n-dimensional intervals in n-space. This will be done in the present paper, independently of the work of Bronowski and Neyman, using the methods of (I). The corresponding problem for circles in the plane will also be solved.

2. n-dimensional intervals, N fixed. Let the random set X be defined as follows. Let A_i, a_i (the range of the subscript i throughout this paper will be from 1 to n) and δ be fixed positive numbers such that $a_i \leq 2\,\delta$. Let R denote the n-dimensional interval consisting of all points $(x_1, \cdots, x_n)$ such that $0 \leq x_i \leq A_i$, and let R' denote the larger interval for which $-\delta \leq x_i \leq A_i + \delta$ (and also its measure $\Pi(A_i + 2\,\delta)$). Let a fixed number N of intervals with sides a_i parallel to the axes be chosen independently, with the probability density function for the center of each interval constant and equal to $1/R'$ in R'. The set X is the intersection of the set-theoretical sum of the N intervals with R. The set Y consists of those points of R that do not belong to X. We have identically

$$X + Y = R, \tag{1}$$

where capital letters denote either sets or their measures.

From (I), equation (15), we have

$$E(Y) = \int_0^{A_n} \cdots \int_0^{A_1} p(x_1, \cdots, x_n)dx_1 \cdots dx_n, \tag{2}$$

where, setting $r = \Pi a_i$, we have

$$p(x_1, \cdots, x_n) = Pr((x_1, \cdots, x_n)\epsilon Y) = \left(1 - \frac{r}{R'}\right)^N. \tag{3}$$

Hence

$$E(Y) = R\left(1 - \frac{r}{R'}\right)^N. \tag{4}$$

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[1] H. E. Robbins. "On the measure of a random set," *Annals of Math. Stat.* Vol. 15 (1944), pp. 70–74. We shall refer to this paper as (I).

[2] J. Bronowski and J. Neyman "On the variance of a random set" *Annals of Math. Stat.* Vol. 16 (1945), pp. 330–341. We shall refer to this paper as (BN).

From (1) it follows that

$$E(X) = R\left\{1 - \left(1 - \frac{r}{R'}\right)^n\right\}. \tag{5}$$

From (I), equation (21), we have

$$E(Y^2) = \int_0^{A_n} \cdots \int_0^{A_1} \int_0^{A_n} \cdots \int_0^{A_1} p(x_1, \cdots x_n, y_1, \cdots, y_n) \cdot dx_1 \cdots dx_n\, dy_1 \cdots dy_n, \tag{6}$$

where

$$p(x_1, \cdots, x_n, y_1, \cdots, y_n) = Pr((x_1, \cdots, x_n) \epsilon Y \text{ and } (y_1, \cdots, y_n) \epsilon Y). \tag{7}$$

It is clear from the symmetry of the problem that the distribution of Y will be unchanged if we assume that for all i, $x_i \le y_i$. Hence, since there are 2^n possible sets of n inequalities each, we can write

$$E(Y^2) = 2^n \int_0^{A_n} \cdots \int_0^{A_1} \int_0^{y_n} \cdots \int_0^{y_1} p\, dx_1 \cdots dx_n\, dy_1 \cdots dy_n. \tag{8}$$

We now introduce the new variables of integration

$$u_i = x_i, \qquad v_i = y_i - x_i \tag{9}$$

for which

$$\frac{\partial(u_1, \cdots, u_n, v_1, \cdots, v_n)}{\partial(x_1, \cdots, x_n, y_1, \cdots, y_n)} = 1. \tag{10}$$

In terms of the new variables we have

$$p = f(v_1, \cdots, v_n) = \begin{cases} \left(1 - \dfrac{2r}{R'}\right)^N \text{ if } v_i \ge a_i \text{ for } \textit{some } i, \\ \left(1 - \dfrac{2r - \Pi(a_i - v_i)}{R'}\right)^N \text{ if } v_i \le a_i \text{ for } \textbf{\textit{all} } i. \end{cases} \tag{11}$$

Equation (8) now becomes

$$\begin{aligned} E(Y^2) &= 2^n \int_0^{A_n} \cdots \int_0^{A_1} \int_0^{A_n - v_n} \cdots \int_0^{A_1 - v_1} f\, du_1 \cdots du_n dv_1 \cdots dv_n \\ &= 2^n \int_0^{A_n} \cdots \int_0^{A_1} f \Pi(A_i - v_i)\, dv_1 \cdots dv_n. \end{aligned} \tag{12}$$

Let $z_i = \min(a_i, A_i)$. Then from (11) and (12) we obtain

$$\begin{aligned} E(Y^2) &= 2^n \int_0^{z_n} \cdots \int_0^{z_1} \left(1 - \frac{2r - \Pi(a_i - v_i)}{R'}\right)^N \Pi(A_i - v_i)\, dv_1 \cdots dv_n \\ &+ 2^n \left(1 - \frac{2r}{R'}\right)^N \left\{\int_0^{A_n} \cdots \int_0^{A_1} \Pi(A_i - v_i)\, dv_1 \cdots dv_n \right. \\ &\left. - \int_0^{z_n} \cdots \int_0^{z_1} \Pi(A_i - v_i)\, dv_1 \cdots dv_n \right\}. \end{aligned} \tag{13}$$

Let the symbol $[x]$, as in (BN), be defined by

$$[x] = \begin{cases} x \text{ if } x \geq 0, \\ 0 \text{ if } x \leq 0. \end{cases} \tag{14}$$

In the integral in the first line of (13) we introduce the new variables of integration $w_i = a_i - v_i$, while in the two integrals in the second line we introduce the variables $s_i = A_i - v_i$. The result is

$$\begin{aligned} E(Y^2) = 2^n \int_{[a_n - A_n]}^{a_n} \cdots \int_{[a_1 - A_1]}^{a_1} \left(1 - \frac{2r - \Pi w_i}{R'}\right)^N & \\ \cdot \Pi(w_i + A_i - a_i)\, dw_1 \cdots dw_n & \\ + 2^n \left(1 - \frac{2r}{R'}\right)^N \left\{ \int_0^{A_n} \cdots \int_0^{A_1} \Pi s_i\, ds_1 \cdots ds_n \right. & \\ \left. - \int_{[A_n - a_n]}^{A_n} \cdots \int_{[A_1 - a_1]}^{A_1} \Pi s_i\, ds_1 \cdots ds_n \right\} & \\ = 2^n \int_{[a_n - A_n]}^{a_n} \cdots \int_{[a_1 - A_1]}^{a_1} \left(1 - \frac{2r - \Pi w_i}{R'}\right)^N & \\ \cdot \Pi(w_i + A_i - a_i)\, dw_1 \cdots dw_n & \\ + \left(1 - \frac{2r}{R'}\right)^N \{\Pi A_i^2 - \Pi(A_i^2 - [A_i - a_i]^2)\}. & \end{aligned} \tag{15}$$

From (1) we see that $\sigma_X^2 = E(X^2) - E^2(X) = E(Y^2) - E^2(Y)$. Thus from (4) and (5) we have

$$\begin{aligned} \sigma_X^2 = 2^n \int_{[a_n - A_n]}^{a_n} \cdots \int_{[a_1 - A_1]}^{a_1} \left(1 - \frac{2r - \Pi w_i}{R'}\right)^N & \\ \cdot \Pi(w_i + A_i - a_i)\, dw_1 \cdots dw_n & \\ + \left(1 - \frac{2r}{R'}\right)^N \{\Pi A_i^2 - \Pi(A_i^2 - [A_i - a_i]^2)\} & \\ - R^2 \left(1 - \frac{r}{R'}\right)^{2N}. & \end{aligned} \tag{16}$$

3. n-dimensional intervals, N variable. Now let X and Y be defined as before except that the number N is taken as a random variable, capable of assuming the values $0, 1, \cdots$ with respective probabilities $p_0, p_1, \cdots$, and with generating function

$$\varphi(t) = \sum_0^\infty p_N t^N. \tag{17}$$

Then from (5) we have

$$E(X) = \sum_0^\infty p_N R\left\{1 - \left(1 - \frac{r}{R'}\right)^N\right\} = R\left\{1 - \varphi\left(1 - \frac{r}{R'}\right)\right\}, \tag{18}$$

while from (15) we have

$$\begin{aligned} \sigma_X^2 = E(Y^2) - E^2(Y) = 2^n \int_{[a_n - A_n]}^{a_n} \cdots \int_{[a_1 - A_1]}^{a_1} \varphi\left(1 - \frac{2r - \Pi w_i}{R'}\right) \\ \Pi(w_i + A_i - a_i)\, dw_1 \cdots dw_n \\ + \varphi\left(1 - \frac{2r}{R'}\right) \{\Pi A_i^2 - \Pi(A_i^2 - [A_i - a_i]^2)\} - R^2 \varphi^2\left(1 - \frac{r}{R'}\right). \end{aligned} \tag{19}$$

In particular, suppose that, as in (BN), N has a Poisson distribution with a parameter λ,

$$p_N = e^{-\lambda R'} \frac{(\lambda R')^N}{N!}, \tag{20}$$

so that

$$\varphi(t) = e^{\lambda R'(t-1)}. \tag{21}$$

Then (18) becomes

$$E(X) = R\{1 - e^{-\lambda r}\}, \tag{22}$$

while (19) becomes

$$\begin{aligned} \sigma_X^2 = 2^n \cdot e^{-2\lambda r} \int_{[a_n - A_n]}^{a_n} \cdots \int_{[a_1 - A_1]}^{a_1} \left\{\sum_0^\infty \frac{(\lambda \Pi w_i)^N}{N!}\right\} \\ \cdot \{\Pi(w_i + A_i - a_i)\}\, dw_1 \cdots dw_n \\ + e^{-2\lambda r} \{\Pi A_i^2 - \Pi(A_i^2 - [A_i - a_i]^2)\} - R^2 e^{-2\lambda r}. \end{aligned} \tag{23}$$

Integrating term by term and simplifying the resulting expression, we obtain finally

$$\begin{aligned} \sigma_X^2 = r \cdot 2^n \cdot e^{-2\lambda r} \sum_1^\infty \left\{\frac{(\lambda r)^N}{N!\{(N+1)(N+2)\}^n} \right. \\ \left. \cdot \Pi\left\{(N+2)A_i - a_i + [a_i - A_i]\left(1 - \frac{A_i}{a_i}\right)^{N+1}\right\}\right\}. \end{aligned} \tag{24}$$

4. Circles in the plane. Let the random set X be defined as follows. Let A_1, A_2, a, and δ be fixed positive numbers such that $2a \leq \min(A_1, A_2, 2\delta)$. Let R denote the rectangle consisting of all points (x_1, x_2) such that $0 \leq x_1 \leq A_1$, $0 \leq x_2 \leq A_2$, and let R' denote the larger rectangle for which $-\delta \leq x_1 \leq A_1 + \delta$, $-\delta \leq x_2 \leq A_2 + \delta$. Let a fixed number N of circles with radii a and areas $b = \pi a^2$ be chosen independently, with the probability density function for

the center of each circle constant and equal to $1/R'$ in R'. The set X is the intersection of the set-theoretical sum of the N circles with R. The set Y consists of those points of R that do not belong to X. Equation (1) holds as before. The analogue of (4) is

$$E(Y) = \int_0^{A_2} \int_0^{A_1} p(x_1, x_2)\, dx_1\, dx_2 = R\left(1 - \frac{b}{R'}\right)^N, \tag{25}$$

while (8) becomes

$$E(Y^2) = 4 \int_0^{A_2} \int_0^{A_1} \int_0^{y_2} \int_0^{y_1} p(x_1, x_2, y_1, y_2)\, dx_1\, dx_2\, dy_1\, dy_2, \tag{26}$$

where

$$p(x_1, x_2, y_1, y_2) = Pr((x_1, x_2)\epsilon Y \text{ and } (y_1, y_2)\epsilon Y). \tag{27}$$

Introducing the new variables (9) we obtain the analogue of (12),

$$E(Y^2) = 4 \int_0^{A_2} \int_0^{A_1} f(A_2 - v_2)(A_1 - v_1)\, dv_1\, dv_2, \tag{28}$$

where, setting $r = (v_1 + v_2)^{\frac{1}{2}}$,

$$f(v_1, v_2) = \begin{cases} \left(1 - \dfrac{2b}{R'}\right)^N \text{ if } r \geq 2a, \\ \left\{1 - \dfrac{2b - 2a^2 \arccos\left(\dfrac{r}{2a}\right) + \dfrac{r}{2}\sqrt{4a^2 - r^2}}{R'}\right\}^N \text{ if } r \leq 2a. \end{cases} \tag{29}$$

Introducing polar coördinates r, θ in the v_1, v_2-plane and carrying out the obvious integrations, we obtain

$$\begin{aligned} E(Y^2) = \left(1 - \frac{2b}{R'}\right)^N &\left\{R^2 + \frac{32}{3} a^3 (A_1 + A_2) - 8a^4 - 4bR\right\} \\ &+ 8a^2 \int_0^1 (\pi R t + 4a^2 t^3 - 4a(A_1 + A_2)t^2) \\ &\cdot \left(1 - \frac{2b - 2a^2 \arccos t + 2a^2\, t\sqrt{1 - t^2}}{R'}\right)^N dt. \end{aligned} \tag{30}$$

If now N is a random variable with generating function (17), then (25) becomes

$$E(Y) = R\varphi\left(1 - \frac{b}{R'}\right), \tag{31}$$

and hence

$$E(X) = R\left\{1 - \varphi\left(1 - \frac{b}{R'}\right)\right\}, \tag{32}$$

while

$$
\begin{aligned}
\sigma_X^2 &= E(X^2) - E^2(X) = E(Y^2) - E^2(Y) \\
&= \varphi\left(1 - \frac{2b}{R'}\right)\left\{R^2 + \frac{32}{3}a^3(A_1 + A_2) - 8a^4 - 4bR\right\} \\
&\qquad - R^2\varphi^2\left(1 - \frac{b}{R'}\right) + 8a^2\int_0^1 (\pi Rt + 4a^2t^3 - 4a(A_1 + A_2)t^2) \\
&\qquad\qquad \cdot \varphi\left(1 - \frac{2b - 2a^2 \arccos t + 2a^2 t\sqrt{1 - t^2}}{R'}\right) dt.
\end{aligned} \tag{33}
$$

SAMPLING FROM A CHANGING POPULATION[1, 2]

By Reinhold Baer

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1. Introduction. If, in sampling a certain population, it is impossible to take more than one sample at any given time, and if the population changes between any two samples, then we are confronted with the following mathematical situation. For every[3] t, $0 \leq t \leq 1$, there is given a distribution[4] (= population) $D(t)$. Let furthermore t_j be, for $0 < j \leq n$, a number between $(j - 1)/n$ and j/n; and assume that x_j is a sample taken from the population $D(t_j)$. We denote by T_n the set of the numbers $t_1, \cdots, t_n$ and by $O(T_n)$ the sample consisting of the x_j; and we assume that $O(T_n)$ is a random sample, i.e. that $x_1, \cdots, x_n$ are independent variables. The question arises to get information concerning the family $D(t)$ from the sample $O(T_n)$. It is clearly hopeless to try for information concerning an individual $D(t)$ or even some $D(t_j)$ or the statistics that may be derived from them. But we may hope for information in the mean, if we assume that the family $D(t)$ is in some sense continuous in t. To make this statement more precise we denote by $a(t)$ the average and by $M_i(t)$ the i-th moment of $D(t)$ around its average. We assume then that $a(t)$ and $M_i(t)$, for $i \leq 8$, exist and are continuous functions of t, and in section 7 we shall have to assume furthermore that $a(t)$ and $M_2(t)$ are functions of bounded variation. These hypotheses assure the existence of

$$\text{the mean average } a = \int_0^1 a(t)\,dt$$

$$\text{and the mean } i\text{-th moment } M_i = \int_0^1 M_i(t)\,dt$$

for $i \leq 8$. Clearly we may hope for information concerning a and M_i from the random sample $O(T_n)$. It is our object to discuss certain more or less well known statistics of the sample $O(T_n)$, and to determine their stochastic limits[5].

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[1] Presented to the American Mathematical Society. September 15, 1945.

[2] The author is indebted to Dr. E. L. Welker for checking the results, in particular those rather obnoxious computations needed in sections 6 and 7 which the author did not incorporate into this paper.

[3] It constitutes a restriction of generality that we consider finite closed intervals only. But it is no further loss in generality to use the interval from 0 to 1, and this choice certainly simplifies notations.

[4] Comparatively little will be assumed of these distributions. These properties will be enumerated in Section 2.

[5] See [2] p 81 and the criterion 2.d. of section 2.

As an illustration we mention the following results which will be obtained in the course of this investigation (among others):[6]

$\bar{x} = n^{-1} \sum_{j=1}^{n} x_j$ converges stochastically to the mean average a;

$s^2 = n^{-1} \sum_{j=1}^{n} (x_j - \bar{x})^2$ converges stochastically to $M_2 + \int_0^1 (a(t) - a)^2 \, dt$,

$d^2 = (2n)^{-1} \sum_{j=1}^{n-1} (x_j - x_{j+1})^2$ converges stochastically to the mean variance M_2.

It is clear that M_2 is the stochastic limit of s^2 if, and only if, $a(t)$ is constant. If $a(t)$ is not constant, then s^2 is not a consistent estimate[7] of M_2, and will have to be rejected—at least for large n—in favor of d^2 which is always a consistent estimate of M_2.

It was this last point that led us into this investigation. Recently the statistic d^2 has found much attention, and the question arose as to why the statistic s^2 should be rejected in favor of d^2. Reading the illuminating introduction of the fundamental paper [1], one sees that just such a situation as we have attempted to describe here in somewhat abstract terms has necessitated the use of d^2. Consequently our result may be considered a theoretical justification for this procedure.

Our other results will be discussed in their interrelation as they are obtained It should be noted that all our results concern themselves with stochastic convergence, and thus they justify the use of a sample function as an estimate of some statistical number only for sufficiently large size n of the sample Thus it is quite possible that for small n other functions provide better estimates. The practical applicability of our results depends, therefore, on a criterion for n to be sufficiently large, and unfortunately such a criterion is not yet available.

2. Notations and fundamental properties. We have not stated in the Introduction the hypotheses to which we subject the distributions under consideration. For our investigation we shall need only very few properties of distributions. Thus we are going to enumerate now some properties of distributions which we are going to use, and we shall assume throughout that these properties are satisfied. As will be seen these hypotheses are rather weak and are satisfied by a large class of distributions.

If x is any stochastic variable, then we denote by $E(x)$ its mathematical expectation, and the only properties of stochastic variables that concern us are properties of their expectations. $E(x)$ is a linear operation satisfying $E(1) = 1$.

[6] It should be noted that the stochastic limit of the following statistics would not be changed, if we substituted for the denominator n of s^2 the denominator $n - 1$ which is often used, and if we allowed the summation in the expression for d^2 to range from 1 to n, defining x_{n+1} as x_1

[7] Wilks [2], p 133.

If furthermore $x_1, \cdots, x_n$ are independent variables, and if the function f depends on some of these variables whereas g depends only on the others, then $E(fg) = E(f)E(g)$, and this property may serve as a definition of independence.

As stated in the Introduction we are going to study a family $D(t)$ of distributions, for $0 \leq t \leq 1$. If x is the stochastic variable of the distribution $D(t)$ for some fixed t, then we let

$$a(t) = E(x) \quad \text{and} \quad M_i(t) = E((x - a(t))^i).$$

We shall assume throughout that the average $a(t)$ and the variance $M_2(t)$ exist for every t, and that $a(t)$ and $M_2(t)$ are continuous functions of t. Moreover, when discussing $M_i(\tau)$, $1 \leq i \leq 4$, we shall assume that every $M_j(\tau)$ with $j \leq 2i$ is a continuous function of τ. Thus we are sure that the mean average a and the mean variance M_2, as defined in the Introduction, always exist, and the mean i-th moment M_i exists, whenever $M_i(t)$ is a continuous function of t.

Remark: If the mean i-th moment M_i exists for every i, then one may be tempted to consider as the mean of the family $D(t)$ a distribution D with average a and i-th moment M_i, provided such a distribution exists. But this has to be done with some caution. For suppose that every $D(t)$ is normal. Then $M_i(t) = 0$ for every odd i, implying $M_i = 0$ for odd i so that D would be symmetric. But $M_{2i}(t) = 1 \cdot 3 \cdots (2i - 1)M_2(t)^i$ and hence $M_{2i} = 1 \cdot 3 \cdots (2i - 1) \cdot \int_0^1 M_2(t)^i \, dt$, and the integral will be the i-th power of M_2 only if $M_2(t)$ is constant. Thus the mean distribution D of a continuous family of normal distributions need not be normal.

As in the Introduction we now let t_i be some number between $(i - 1)/n$ and i/n, and denote by x_i a sample taken from the distribution $D(t_i)$. We denote by T_n the set of the n numbers t_i and by $O(T_n)$ the sample consisting of the x_i. It will be assumed throughout that $O(T_n)$ is a random sample, i.e. we shall assume that $x_1, \cdots, x_n$ are independent variables.

We are not going to make any use of the customary definition of stochastic convergence[8] (and we shall therefore not restate it). Instead we are going to apply throughout the following criterion[9, 10]:

2.d. *The function $f(O(T_n))$ of the sample $O(T_n)$ converges stochastically to the number r, if*

$$\lim_{n\to\infty} E(f(O(T_n))) = r \text{ and } \lim_{n\to\infty} E([f(O(T_n)) - E(f(O(T_n)))]^2) = 0.$$

All the sample functions considered will be polynomials of the variables $x_1, \cdots, x_n$.

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[8] Wilks [2], p. 81.

[9] Wilks [2], Theorem (A), p 134.

[10] The validity of criterion 2.d. implies stochastic convergence in the customary sense. Thus, all results obtained in the present paper remain valid also when the customary definition of stochastic convergence is adopted.

3. The mean average. Though the discussion of this section is rather obvious, we give the details, since they may serve as a convenient introduction to the type of argument we have to use throughout.

THEOREM. *$\bar{x}$ converges stochastically to a.*

PROOF: We note first that $E(\bar{x}) = n^{-1} \sum_{j=1}^{n} E(x_j) = n^{-1} \sum_{j=1}^{n} a(t_j)$ Since t_j is between $(j-1)/n$ and j/n, and since n^{-1} is the length of this interval, it follows from the continuity of $a(t)$ that

$$\int_0^1 a(t)\,dt = \lim_{n\to\infty} n^{-1} \sum_{j=1}^{n} a(t_j);$$

and thus we have shown that $E(\bar{x})$ tends to a as n tends to infinity.

Next we find that

$$E((\bar{x} - E(\bar{x}))^2) = n^{-2} E\left(\left[\sum_{j=1}^{n} (x_j - a(t_j))\right]^2\right)$$
$$= n^{-2} \sum_{j=1}^{n} E((x_j - a(t_j))^2) = n^{-2} \sum_{j=1}^{n} M_2(t_j),$$

since $E((x_j - a(t_j))(x_h - a(t_h))) = E(x_j - a(t_j))E(x_h - a(t_h)) = 0$ for $j \neq h$ But $M_2(t)$ is, for $0 \leq t \leq 1$, a bounded non-negative function, showing that $E((\bar{x} - E(\bar{x}))^2)$ tends to 0 as n tends to infinity. Applying 2.d. we find that $\bar{x}$ converges stochastically to a, as we intended to show.

Remark· It is clear that the speed of the stochastic convergence of $\bar{x}$ to a depends on two factors:

(i) the goodness of $\bar{x}$ as an estimate of $E(\bar{x})$;

(ii) the speed of convergence of the sums $n^{-1} \sum_{j=1}^{n} a(t_j)$ to the integral $a = \int_0^1 a(t)\,dt$.

It is this difficulty which expresses itself in (ii) and which makes the present type of statistical estimation less effective than the one concerned with sampling from one distribution only. As to (i), it is again, as may be seen from the proof, of the order of magnitude $(M_2/n)^{\frac{1}{2}}$, (see Theorem 1, section 4)

It is probable that $\bar{x}$ is a better estimate of $E(\bar{x})$ than of a But this does not help, since the former depends on the particular choice of T_n

4. The variance. THEOREM 1 *d^2 converges stochastically to M_2.*

PROOF: We note first that

$$E((x_j - x_{j+1})^2 = E([(x_j - a(t_j)) + (a(t_j) - a(t_{j+1})) + (a(t_{j+1}) - x_{j+1})]^2)$$
$$= M_2(t_j) + (a(t_j) - a(t_{j+1}))^2 + M_2(t_{j+1}),$$

since $E((x_j - a(t_j))(x_{j+1} - a(t_{j+1}))) = E(x_j - a(t_j))E(x_{j+1} - a(t_{j+1})) = 0$, $E(\text{const}) = \text{const}$ and $E((x_i - a(t_i))^2) = M_2(t_i)$. Hence

$$E(d^2) = (2n)^{-1}(A + B - C),$$

where $A = 2\sum_{j=1}^{n} M_2(t_j)$, $B = \sum_{j=1}^{n-1} (a(t_j) - a(t_{j+1}))^2$, $C = M_2(t_1) + M_2(t_n)$. Since t_j is a value between $(j-1)/n$ and j/n, and since n^{-1} is the length of this interval, it follows from the continuity of the function $M_2(t)$ that $M_2 = \int_0^1 M_2(t)\,dt = \lim_{n\to\infty} (2n)^{-1}A$. Since $M_2(t)$ is bounded as a continuous function, it follows that $(2n)^{-1}C$ tends to 0 as n tends to infinity. Finally we infer from the continuity of $a(t)$ —which is used here for the first time to its full extent—that there exists to every given positive ϵ an integer $N = N(\epsilon)$ such that $(a(t') - a(t''))^2 < \epsilon$ for $|t' - t''| < (2N)^{-1}$. Thus for $N(\epsilon) < n$ we have $(a(t_j) - a(t_{j+1}))^2 < \epsilon$ and $(2n)^{-1}B < n\frac{1}{2n}\epsilon$. Hence $(2n)^{-1}B$ tends to 0 as n tends to infinity, and we have shown that

$E(d^2)$ tends to M_2 as n tends to infinity.

Next we note that

$$E((d^2 - E(d^2))^2) = E(d^4) - E(d^2)^2$$
$$= (2n)^{-2}\sum_{i,j}[E((x_i - x_{i+1})^2(x_j - x_{j+1})^2) - E((x_i - x_{i+1})^2)E((x_j - x_{j+1})^2)].$$

But if both i and $i+1$ are different from j and $j+1$, then $E((x_i - x_{i+1})^2(x_j - x_{j+1})^2) = E((x_i - x_{i+1})^2)E((x_j - x_{j+1})^2)$, and thus there are not more than $3n$ summands in the above summation that are not identically 0. These summands, however, depend only on $a(t_k)$, $M_2(t_k)$, $M_3(t_k)$ and $M_4(t_k)$, and they are therefore bounded. Thus $E((d^2 - E(d^2))^2)$ is equal to $(2n)^{-2}$ times a sum of not more than $3n$ summands which are bounded. Hence $E((d^2 - E(d^2))^2)$ tends to 0, as n tends to infinity. Now our theorem is an immediate consequence of the criterion 2.d.

THEOREM 2. *s^2 converges stochastically to $M_2 + \int_0^1 (a(t) - a)^2\,dt$.*

PROOF: We note first that $n(x_j - \bar{x}) = \sum_{h=1}^{n}(x_j - x_h)$ and that therefore $s^2 = n^{-3}\sum_{j=1}^{n}\sum_{h,k}(x_j - x_h)(x_j - x_k)$. Since $x_i - x_j = x_i - a(t_i) + a(t_i) - a(t_j) - (x_j - a(t_j))$, we find as usual that

$$E((x_j - x_h)^2) = M_2(t_j) + (a(t_j) - a(t_h))^2 + M_2(t_h),$$

and if $h \neq k$ we find that

$$E((x_j - x_h)(x_j - x_k)) = M_2(t_j) + (a(t_j) - a(t_h))(a(t_j) - a(t_k)).$$

Consequently

$$\sum_{h,k} E((x_j - x_h)(x_j - x_k)) = n^2 M_2(t_j) + \sum_{h=1}^{n} M_2(t_h)$$
$$+ \sum_{h,k}(a(t_j) - a(t_h))(a(t_j) - a(t_k))$$
$$= n^2 M_2(t_j) + \sum_{h=1}^{n} M_2(t_h) + \left[\sum_{h=1}^{n}(a(t_j) - a(t_h))\right]^2.$$

Consequently

$$E(s^2) = n^{-1}\sum_{j=1}^{n} M_2(t_j) + n^{-2}\sum_{h=1}^{n} M_2(t_h) + n^{-3}\sum_{j=1}^{n}\left[\sum_{h=1}^{n}(a(t_j) - a(t_h))\right]^2$$

As in the proof of Theorem 1 we see that the first of these sums tends to M_2 as n tends to infinity, and the second of these sums therefore tends to 0 as n tends to infinity. The last sum equals

$$n^{-3}\sum_{j,h,k}[a(t_j)^2 - a(t_j)(a(t_h) + a(t_k)) + a(t_h)a(t_k)]$$

$$= n^{-1}\sum_{j=1}^{n} a(t_j)^2 - 2n^{-2}\sum_{j,h} a(t_j)a(t_h) + n^{-2}\sum_{h,k} a(t_h)a(t_k)$$

$$= n^{-1}\sum_{j=1}^{n} a(t_j)^2 - \left[n^{-1}\sum_{j=1}^{n} a(t_j)\right]^2,$$

and this expression tends to $\int_0^1 a(t)^2\,dt - \left[\int_0^1 a(t)\,dt\right]^2$ as n tends to infinity. But

$$\int_0^1 a(t)^2\,dt - \left[\int_0^1 a(t)\,dt\right]^2 = \int_0^1 (a(t) - a)^2\,dt,$$

since $a = \int_0^1 a(t)\,dt$, and thus we have shown that $E(s^2)$ tends to $M_2 + \int_0^1 (a(t) - a)^2\,dt$ as n tends to infinity.

If j, h, k, p, q, r are integers between 1 and n, we put

$$(j, h, k, p, q, r) = E((x_j - x_h)(x_j - x_k)(x_p - x_q)(x_p - x_r)) - E((x_j - x_h)(x_j - x_k))E((x_p - x_q)(x_p - x_r)).$$

If neither j, h nor k is equal to any of the three integers p, q, r, it follows from the independence of the variables x_i that $(j, h, k; p, q, r) = 0$. Thus

$$E((s^2 - E(s^2))^2) = E(s^4) - E(s^2)^2 = n^{-6}\,\Sigma'(j, h, k, p, q, r),$$

where the summation is taken over all the values of j, h, k, p, q, r between 1 and n with the restriction that at least one of the three numbers j, h, k is equal to at least one of the three numbers p, q, r. This sum contains therefore not more than 3^3n^5 summands, and each of the summands is bounded, since they depend only on $a(t_i)$, $M_2(t_i)$, $M_3(t_i)$ and $M_4(t_i)$. Thus $E((s^2 - E(s^2))^2)$ is equal to n^{-6} times a sum of not more than 3^3n^5 summands which are bounded. Hence $E((s^2 - E(s^2))^2)$ tends to 0 as n tends to infinity. Now our theorem is an immediate consequence of the criterion 2.d.

Noting that $\int_0^1 (a(t) - a)^2\,dt$ is nothing but the variance of the function $a(t)$ (around its mean a), we obtain the following obvious consequence of Theorems 1 and 2.

COROLLARY: *$s^2 - d^2$ converges stochastically to the variance of $a(t)$.*

Remarks similar to those made in connection with the proof of the theorem of section 3 may be made now in regard to the theorems of this section.

By similar arguments it is possible to prove that the statistic $n^{-1}\sum_{i=1}^{n-1} x_i x_{i+1}$ converges stochastically to $\int_0^1 a(t)^2\,dt$.

5. The third moment. Put $d(3) = n^{-1}\sum_{j=1}^{n-2}(x_j - x_{j+1})^2(x_{j+1} - x_{j+2})$. Then $d(3)$ is a function of the random sample $O(T_n)$.

THEOREM 1: *$d(3)$ converges stochastically to M_3.*

PROOF: It is readily seen that

$$E((x_j - x_{j+1})^2(x_{j+1} - x_{j+2})) = M_3(t_{j+1}) + (a(t_{j+1}) - a(t_{j+2}))(M_2(t_j) + (a(t_j) - (a(t_j) - a(t_{j+1}))^2 + M_2(t_{j+1})),$$

and in practically the same fashion as in the proof of Theorem 1 of section 4 one shows now that $E(d\,(3))$ tends to M_3 as n tends to infinity.

Furthermore we have

$$E((d(3) - E(d(3))^2) = E(d(3)^2) - E(d(3))^2 = n^{-2}\sum_{j,h}(j, h),$$

where

$$(j, h) = E((x_j - x_{j+1})^2(x_{j+1} - x_{j+2})(x_h - x_{h+1})^2(x_{h+1} - x_{h+2})) - E((x_j - x_{j+1})^2(x_{j+1} - x_{j+2}))E((x_h - x_{h+1})^2(x_{h+1} - x_{h+2})).$$

Clearly $(j, h) = 0$ whenever $j + 2 < h$ or $h + 2 < j$. Consequently there appear actually in the sum of all the (j, h) not more than $5n$ terms each of which is bounded by an absolute constant, since they depend only on $a(t_i)$, $M_2(t_i)$, $M_3(t_i)$, $M_4(t_i)$, $M_5(t_i)$ and $M_6(t_i)$. From this fact we infer as before that $E((d(3) - E(d(3))^2)$ tends to 0, as n tends to infinity, and our theorem is an immediate consequence of the criterion 2.d.

Remark 1. If $M_3(t)$, $M_2(t)$ and $a(t)$ are constant, it follows from the proof that

$$E(d(3)) = \frac{n-2}{n} M_3\,;$$

and thus $(n-2)^{-1}\sum_{j=1}^{n-2}(x_j - x_{j+1})^2(x_{j+1} - x_{j+2})$ is an unbiased estimate of M_3.

Remark 2. One might be tempted to use instead of $d(3)$ the following function:

$$n^{-1}\sum_{j=1}^{n-1}(x_j - x_{j+1})^3.$$

By an argument of a nature rather similar to the one used in the preceding proof one may show, however, that this statistic converges stochastically to 0.

Put $s(3) = n^{-1}\sum_{j=1}^{n}(x_j - \bar{x})^3$. Then $s(3)$ is a function of the random sample $O(T_n)$. Furthermore let

$$F_3 = 3\left(\int_0^1 a(t)M_2(t)\,dt - aM_2 - a\int_0^1 a^2(t)\,dt\right) + 2a^3 + \int_0^1 a^3(t)\,dt.$$

THEOREM 2. *$s(3)$ converges stochastically to $M_3 + F_3$.*

PROOF: For fixed j, let $X(j) = \sum_{h=1}^{n}(x_j - a(t_j) + a(t_h) - x_h)$ and $A(j) = \sum_{h=1}^{n}(a(t_j) - a(t_h))$. Then

$$E(s(3)) = n^{-4}\sum_{j=1}^{n}E((X(j) + A(j))^3)$$
$$= n^{-4}\sum_{j=1}^{n}[E(X(j)^3) + 3A(j)E(X(j)^2) + A(j)^3],$$

since $E(X(j))$ is easily seen to be 0. We find furthermore that

$$E(X(j)^3) = (n-1)^3M_3(t_j) + E([\sum_{h\neq j}(a(t_h) - x_h)]^3)$$
$$= ((n-1)^3 + 1)M_3(t_j) - \sum_{h=1}^{n}M_3(t_h);$$

$$E(X(j)^2) = (n-1)^2M_2(t_j) + E(\sum_{h\neq 1}(a(t_h) - x_h)^2)$$
$$= ((n-1)^2 - 1)M_2(t_j) + \sum_{h=1}^{n}M_2(t_h).$$

Consequently

$$E(s(3)) = n^{-4}\Bigg[((n-1)^3 - n + 1)\sum_{j=1}^{n}M_3(t_j) + 3((n-1)^2 - 1)\sum_{j=1}^{n}A(j)M_2(t_j)$$
$$+ 3\sum_{j=1}^{n}A(j)\sum_{h=1}^{n}M_2(t_h) + \sum_{j=1}^{n}A(j)^3\Bigg].$$

Since furthermore $\sum_{j=1}^{n}A(j) = \sum_{j,h}(a(t_j) - a(t_h)) = 0$,

$$\sum_{j=1}^{n}A(j)M_2(t_j) = n\sum_{j=1}^{n}a(t_j)M_2(t_j) - \sum_{h=1}^{n}a(t_h)\sum_{j=1}^{n}M_2(t_j)$$

and

$$\sum_{j=1}^{n}A(j)^3 = \sum_{j=1}^{n}\left[na(t_j) - \sum_{h=1}^{n}a(t_h)\right]^3$$
$$= n^3\sum_{j=1}^{n}a(t_j)^3 - 3n^2\sum_{j=1}^{n}a(t_j)^2\sum_{h=1}^{n}a(t_h) + 3n\sum_{j=1}^{n}a(t_j)\left[\sum_{h=1}^{n}a(t_h)\right]^2$$
$$- n\left[\sum_{h=1}^{n}a(t_h)\right]^3,$$

it is easily verified that $E(s(3))$ tends to $M_3 + F_3$, as n tends to infinity.

To prove that $E((s(3) - E(s(3)))^2)$ tends to 0 as n tends to infinity, one proceeds as in the proofs of the preceding theorems, namely by verifying that this expectation is n^{-8} times a sum of not more than $4^4 n^7$ summands which are bounded, since they depend only on $a(t_i)$ and on the $M_m(t_i)$ for $1 < m < 7$. The proof of the theorem may then be completed by applying the criterion 2.d

It is readily seen that F_3 vanishes whenever $a(t)$ is constant. But from

$$F_3 = 3\left[\int_0^1 a(t)M_2(t)\,dt - aM_2\right] + \int_0^1 (a(t) - a)^3\,dt$$

we infer that F_3 vanishes too whenever $M_2(t)$ is constant and $a(t)$ is at the same time symmetric with regard to a, and more precisely: if $M_2(t)$ is constant, a necessary and sufficient condition for the vanishing of F_3 is the vanishing of the third moment of the function $a(t)$ around its mean. Thus we see that $d(3)$ is always a consistent statistic for M_3, though $s(3)$ is not.

6. The fourth moment. The results in this section will be stated without proof. Their proofs can be constructed on exactly the same lines as the proofs in sections 4 and 5.

$$(2n)^{-1}\sum_{j=1}^{n-1}(x_j - x_{j+1})^4, \qquad n^{-1}\sum_{j=2}^{n-1}(x_{j-1} - x_j)^3(x_{j+1} - x_j)$$

and

$$n^{-1}\sum_{j=2}^{n-1}(x_{j-1} - x_j)^2(x_{j+1} - x_j)^2$$

converge stochastically to $M_4 + 3\int_0^1 M_2(t)^2\,dt$.

$(4n)^{-1}\left[\sum_{j=1}^{n-1}(x_j - x_{j+1})^2\right]^2$ *converges stochastically to* $M_4 + M_2^2$.

$(4n)^{-1}\sum_{j=2}^{n-2}(x_{j-1} - x_j)^2(x_{j+1} - x_{j+2})^2$ *converges stochastically to* $\int_0^1 M_2(t)^2\,dt$.

From these facts one easily deduces that M_4 *is the stochastic limit of*

$$n^{-1}\left[\frac{1}{2}\sum_{j=1}^{n-1}(x_j - x_{j+1})^4 - \frac{3}{4}\sum_{j=2}^{n-2}(x_{j-1} - x_j)^2(x_{j+1} - x_{j+2})^2\right],$$

and that $\int_0^1 (M_2(t) - M^2)^2\,dt$ *is the stochastic limit of*

$$(2n)^{-1}\left[\sum_{j=1}^{n-1}(x_j - x_{j+1})^4 - \frac{1}{2}\left\{\sum_{j=1}^{n-1}(x_j - x_{j+1})^2\right\}^2 - \sum_{j=2}^{n-2}(x_{j-1} - x_j)^2(x_{j+1} - x_{j+2})^2\right].$$

7. Efficiency. If $f = f(O(T_n))$ is a function of the random sample $O(T_n)$, and if f converges stochastically to a number r, then

$$\lim_{n\to\infty} nE((f - r)^2)$$

may be considered as some sort of a measure for the efficiency[11, 12] of the statistic f as an estimate of r, provided, of course, the limit exists.

THEOREM 1 *If the function $a(t)$ is of bounded variation, then*

$$\lim_{n\to\infty} nE((\bar{x} - a)^2) = M_2 .$$

PROOF: Clearly

$$nE((\bar{x} - a)^2) = n^{-1}E\left(\left[\sum_{j=1}^{n} (x_j - a)\right]^2\right)$$

$$= n^{-1}\sum_{j=1}^{n} M_2(t_j) + n^{-1}\left[\sum_{j=1}^{n} (a(t_j) - a)\right]^2$$

$$\text{Now } \sum_{j=1}^{n} (a(t_j) - a) = \sum_{j=1}^{n} a(t_j) - na = \sum_{j=1}^{n}\left[a(t_j) - n\int_{(j-1)/n}^{j/n} a(t)dt\right]$$

Since $a(t)$ is a continuous function, there exists a number u_j such that

$$(j - 1)/n \leq u_j \leq j/n, \quad \text{and} \quad \int_{(j-1)/n}^{j/n} a(t)dt = n^{-1} a(u_j)$$

Thus

$$\sum_{j=1}^{n} (a(t_j) - a) = \sum_{j=1}^{n} (a(t_j) - a(u_j))$$

But both t_j and u_j are between $(j - 1)/n$ and j/n, and $a(t)$ is of bounded variation. Hence there exists a constant A which depends on $a(t)$ only and not on n or T_n such that

$$\left[\sum_{j=1}^{n} (a(t_j) - a)\right]^2 \leq A \text{ for every choice of } T_n.$$

The contention of our theorem is a fairly immediate consequence of these facts.

This theorem and its proof may serve as an additional substantiation of the remarks appended to section 3.

Remark: If we had assumed only the continuity of $a(t)$ instead of its being of bounded variation, we could have tried to argue as follows: Since $a(t)$ is continuous, there exists to every positive number ϵ an integer $N(\epsilon)$ such that $| a(t') - a(t'') | < \epsilon$ for $| t' - t'' | < N(\epsilon)^{-1}$ Hence we would find that for $N(\epsilon) < n$ we have

$$n^{-1}\left[\sum_{j=1}^{n} (x_j - a)\right]^2 < n\epsilon^2,$$

and this inequality is certainly insufficient for proving that the left side of the inequality tends to 0 as n tends to infinity.

THEOREM 2: *If the functions $a(t)$ and $M_2(t)$ are both of bounded variation, then* $\lim_{n\to\infty} nE((d^2 - M_2)^2) = M_4$.

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[11] Wilks [2], p 134/135.

[12] or a measure for the asymptotic variance of the function f

PROOF: In the course of the proof of Theorem 1 of section 4 we have shown that $E(d^2) = (2n)^{-1}(A + B - C)$, where

$$A = 2\sum_{j=1}^{n} M_2(t_j), B = \sum_{j=1}^{n-1} (a(t_j) - a(t_{j+1}))^2, C = M_2(t_1) + M_2(t_n).$$

Since $M_2(t)$ is bounded, it is clear that $n^{-1}C$ tends to 0 as n tends to infinity. Since $a(t)$ is of bounded variation, there exists a constant B^* such that $B \leq B^*$ for every choice of T_n, and hence $n^{-1}B$ tends to 0 as n tends to infinity.[13] Furthermore we have

$$\sum_{j=1}^{n} M_2(t_j) - nM_2 \sum_{j=1}^{n} \left[M_2(t_j) - n\int_{(j-1)/n}^{j/n} M_2(t)dt\right].$$

Because of the continuity of $M_2(t)$ there exist numbers v_j such that

$$(j-1)/n \leq v_j \leq j/n, \quad \text{and} \quad M_2(v_j) = n\int_{(j-1)/n}^{j/n} M_2(t)dt.$$

Consequently

$$\sum_{j=1}^{n} M_2(t_j) - nM_2 = \sum_{j=1}^{n} [M_2(t_j) - M_2(v_j)].$$

But $M_2(t)$ is a function of bounded variation, and thus we may infer, as in the proof of Theorem 1, that $n^{\frac{1}{2}}[(2n)^{-1}A - M_2]$ tends to 0 as n tends to infinity. Combining all the facts we see that $n^{\frac{1}{2}}[E(d^2) - M_2]$ tends to 0 as n tends to infinity, and hence we have shown that $n[E(d^2) - M_2]^2$ tends to 0, as n tends to infinity.

As in the proof of Theorem 1 of section 4 we note next that

$$E(d^4) - E(d^2)^2 = (2n)^{-2}\sum_{i,j} (i,j),$$

where $(i, j) = E((x_i - x_{i+1})^2(x_j - x_{j+1})^2) - E((x_i - x_{i+1})^2)E((x_j - x_{j+1})^2)$, and that $(i, j) = 0$, if either $i + 1 < j$ or $j + 1 < i$. Next we observe that

$$\begin{aligned}(i, j) = {} & E((x_i - a(t_i) + a(t_{i+1}) - x_{i+1})^2(x_j - a(t_j) + a(t_{j+1}) - x_{j+1})^2) \\ & - E((x_i - a(t_i) + a(t_{i+1}) - x_{i+1})^2)E((x_j - a(t_j) + a(t_{j+1}) - x_{j+1})^2) \\ & + (a(t_i) - a(t_{i+1}))(i, j)' + (a(t_j) - a(t_{j+1}))(i, j)'',\end{aligned}$$

where the expressions $(i, j)'$ and $(i, j)''$ are bounded (by a number independent of i, j, n or T).

Consequently we have

$$\begin{aligned}(i, i) &= M_4(t_i) + 6M_2(t_i)M_2(t_{i+1}) + M_4(t_{i+1}) - (M_2(t_i) + M_2(t_{i+1}))^2 \\ &\qquad + (a(t_i) - a(t_{i+1}))(i, i)^* \\ &= M_4(t_i) + M_4(t_{i+1}) + M_2(t_i)^2 + M_2(t_{i+1})^2 \\ &\qquad - 2(M_2(t_i) - M_2(t_{i+1}))^2 + (a(t_i) - a(t_{i+1}))(i, i)^*,\end{aligned}$$

where $(i, i)^* = (i, i)' + (i, i)''$ is bounded by a bound independent of i, n, T_n.

[13] A remark similar to the one made just before stating Theorem 2 may be made here and below about the indispensability of the hypothesis that $a(t)$ and $M_2(t)$ be of bounded variation.

Likewise we find that

$$\begin{aligned}(i, i+1) &= M_2(t_i)M_2(t_{i+1}) + M_2(t_i)M_2(t_{i+2}) + M_4(t_{i+1}) + M_2(t_{i+1})M_2(t_{i+2}) \\ &\quad - (M_2(t_i) + M_2(t_{i+1}))(M_2(t_{i+1}) + M_2(t_{i+2})) \\ &\quad + (a(t_i) - a(t_{i+1}))(i, i+1)' + (a(t_{i+1}) - a(t_{i+2}))(i, i+1)'' \\ &= M_4(t_{i+1}) - M_2(t_{i+1})^2 \\ &\quad + (a(t_i) - a(t_{i+1}))(i, i+1)' + (a(t_{i+1}) - a(t_{i+2}))(i, i+1)''\end{aligned}$$

Hence

$$\begin{aligned}(i, i) + 2(i, i+1) &= M_4(t_i) + 3M_4(t_{i+1}) + (M_2(t_i) \\ &\quad - M_2(t_{i+1}))(3M_2(t_{i+1}) - M_2(t_1)) + (a(t_i) \\ &\quad - a(t_{i+1}))(i, i)^+ + (a(t_{i+1}) \\ &\quad - a(t_{i+2}))(i, i+1)'',\end{aligned}$$

where $(i, i)^+ = (i, i)' + (i, i)'' + (i, i+1)'$ is bounded by a bound independent of i, n, T. Considering that

$$\sum_{i,j} (i,j) = \sum_{i=1}^{n-1} (i, i) + 2\sum_{i=1}^{n-2} (i, i+1),$$

it is now deduced from the continuity of the functions $a(t)$, $M_2(t)$ and $M_4(t)$ that $n[E(d^4) - E(d^2)^2]$ tends to M_4, as n tends to infinity. We note finally that $E((d^2 - M_2)^2) = E((d^2 - E(d^2))^2) + (E(d^2) - M_2)^2$, and the theorem is an immediate consequence of the facts we have deduced.

THEOREM 3. *If the functions $a(t)$ and $M_2(t)$ are both of bounded variation, then*

$$\begin{aligned}&\lim_{n\to\infty} nE((s^2 - M_2 - \int_0^1 (a(t) - a)^2\,dt)^2) \\ &= M_4 - \int_0^1 M_2(t)^2\,dt + 4\int_0^1 (a(t)M_3(t) - aM_3)\,dt + 4\int_0^1 M_2(t)(a(t) - a)^2\,dt.\end{aligned}$$

PROOF Since $a(t)$ and $M_2(t)$ are of bounded variation, we show—as in the proofs of the two preceding theorems—that

$$n^{\frac12}(n^{-1}\sum_{j=1}^{n} a(t_j) - a), n^{\frac12}(n^{-1}\sum_{j=1}^{n} a(t_j)^2 - \int_0^1 a(t)^2\,dt), \quad \text{and}$$

$$n^{\frac12}(n^{-1}\sum_{j=1}^{n} M_2(t_j) - M_2)$$

all tend to 0, as n tends to infinity In the proof of Theorem 2 of section 4 we computed $E(s^2)$. Using this result we obtain:

$$\begin{aligned}&n^{\frac12}(E(s^2) - M_2\int_0^1 (a(t) - a)^2\,dt) \\ &\quad = n^{\frac12}(n^{-1}\sum_{j=1}^{n} M_2(t_j) - M_2) + n^{-\frac12}n^{-1}\sum_{j=1}^{n} M_2(t_j) \\ &\qquad + n^{\frac12}(n^{-1}\sum_{j=1}^{n} a(t_j)^2 - \int_0^1 a(t)^2\,dt) \\ &\qquad + n^{\frac12}\left(a^2 - \left[n^{-1}\sum_{j=1}^{n} a(t_j)\right]^2\right),\end{aligned}$$

where one should remember the identity $\int_0^1 (a(t) - a)^2\,dt = \int_0^1 a(t)^2\,dt - a^2$.
But

$$n^{\frac{1}{2}}\left(a^2 - \left[n^{-1}\sum_{j=1}^{n} a(t_j)\right]^2\right) = n^{\frac{1}{2}}\left(a - n^{-1}\sum_{j=1}^{n} a(t_j)\right)\left(a + n^{-1}\sum_{j=1}^{n} a(t_j)\right),$$

where the last factor on the right is bounded by a bound independent of n and T_n. Hence it follows that

$$n\left(E(s^2) - M_2 - \int_0^1 (a(t) - a)^2\,dt\right)^2 \text{ tends to 0, as } n \text{ tends to infinity.}$$

By a computation of great length and little interest one shows that

$$\begin{aligned} nE((s^2 - E(s^2))^2) = n^{-3}\Bigg[&(n-1)^2\sum_{j=1}^{n} M_4(t_j) + 4n(n-1)\sum_{j=1}^{n} M_3(t_j)a(t_j) \\ &- 4(n-1)\sum_{j=1}^{n} M_3(t_j)\sum_{h=1}^{n} a(t_h) + 2\left[\sum_{j=1}^{n} M_2(t_j)\right]^2 \\ &- (n^2 - 2n + 3)\sum_{j=1}^{n} M_2(t_j)^2 + 4n^2\sum_{j=1}^{n} M_2(t_j)a(t_j)^2 \\ &- 8n\sum_{j=1}^{n} a(t_j)\sum_{h=1}^{n} a(t_h)M_2(t_h) \\ &+ 4\left[\sum_{j=1}^{n} a(t_j)\right]^2\sum_{h=1}^{n} M_2(t_h)\Bigg]. \end{aligned}$$

It is readily seen that this expression tends to

$$\begin{aligned} M_4 + 4\int_0^1 M_3(t)a(t)\,dt - 4M_3 a - \int_0^1 M_2(t)^2\,dt + 4\int_0^1 M_2(t)a(t)^2\,dt \\ - 8a\int_0^1 a(t)M_2(t)\,dt + 4a^2M_2, \end{aligned}$$

and now it is clear how to complete the proof of our theorem.

Corollary 1 *If $a(t)$ is constant and $M_2(t)$ of bounded variation, then* $\lim_{n\to\infty} nE((s^2 - M_2)^2) = M_4 - \int_0^1 M_2(t)^2\,dt.$

This is an almost immediate consequence of Theorem 3, since $a(t) = a$, if $a(t)$ is constant.

It has been shown in section 4 that d^2 is always a consistent estimate of M_2 whereas s^2 is a consistent estimate of M_2 if, and only if, $a(t)$ is constant. Theorem 1 and Corollary 1 offer a basis for comparing the efficiency of these two statistics. Since

$$0 < M_2(t)^2 < M_4(t) \text{ for every } t$$

(apart from trivial exceptions), we infer from Theorem 1 and Corollary 1 the following fact.

COROLLARY 2. *If $a(t)$ is constant and $M_2(t)$ of bounded variation, then*

$$\lim_{n\to\infty} \frac{E((s^2 - M_2)^2)}{E((d^2 - M_2)^2)} = 1 - \frac{\int_0^1 M_2(t)^2\,dt}{M_4},$$

and this expression is always positive and smaller than 1.

Thus we may say roughly that for large n the estimate s^2 of M_2 is more efficient than the estimate d^2, in case both may be used[14] We do, however, not offer any information of the necessary size of n. Neither do we claim that for small n it might not happen that d^2 gives a good estimate and s^2 a poor one.

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[14] It has been pointed out before that s^2 is a consistent estimate of M_2 if, and only if, $a(t)$ is constant, and thus the efficiency of s^2 and d^2 as estimates of M_2 may be compared only if $a(t)$ is constant.

TESTING THE HOMOGENEITY OF POISSON FREQUENCIES

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1. Introduction. The standard procedure for testing the homogeneity of a set of k Poisson frequencies seems to be to apply the Poisson index of dispersion to those frequencies. The originators of this procedure [1] pointed out that this procedure may be regarded as a χ^2 test of goodness of fit in which the Poisson frequencies constitute observed frequencies corresponding to k cells with equal expected values. Somewhat later it was shown [2] that the corresponding likelihood ratio test was approximately equivalent to the index of dispersion test. Then the problem was approached from the viewpoint of conditional variation [3], [4]. This approach permitted exact tests to be studied in some detail for small samples. A few years later an exact test for the special case of $k = 2$ was introduced and studied [5]. In this investigation consideration was given for the first time to the efficiency of the proposed test. Tables of critical regions for the test and tables for computing the power of the test corresponding to certain alternatives were made available.

In spite of the desirable features of this last test, it still possesses certain drawbacks. First, this test, as well as the others referred to, did not consider the problem in which the rate of occurrence of a rare event is constant but for which the sampling units differ in size. For example, these methods were not designed to enable one to test whether a factory's accident rate had remained unchanged during the past month as compared with the preceding three months. Second, in order to use this test it is necessary to possess the special tables or charts of critical regions constructed for the test.

In this paper a method which does not require special tables is considered for dealing with these more general situations. In the course of the development it is shown that this method is, in a certain sense, the best method possible for testing the hypothesis of homogeneity against one sided alternatives. Since this paper is principally concerned with removing the undesirable features of the method advocated in the last mentioned paper, it is advisable to read that paper in conjunction with this one. The procedure to be followed here will be to derive a uniformly most powerful test, show that it is equivalent to a χ^2 test, and then compare it with the previously mentioned test.

2. Similar regions. In the following two sections a study will be made of the efficiency of a generalization of the critical region proposed in [5]. For this purpose let x and y represent sample frequencies from two independent Poisson distributions with means m_x and m_y. The probability of obtaining this sample is given by

$$P(x, y) = \frac{e^{-m_x} m_x^x}{x!} \cdot \frac{e^{-m_y} m_y^y}{y!}. \tag{1}$$

Following the notation and procedure given in [5], let

$$\mu = m_x + m_y, \qquad p = \frac{m_x}{m_x + m_y}, \qquad n = x + y \tag{2}$$

Then algebraic manipulation will show that $P(x, y)$ reduces to

$$P(x, y) = \frac{e^{-\mu}\mu^n}{n!} \cdot \frac{n!}{x!(n-x)!} p^x(1-p)^{n-x} \tag{3}$$

The hypothesis which it is desired to test is that

$$\frac{m_y}{m_x} = r, \tag{4}$$

where r has been specified. The value of r will often be the ratio of the sizes of the two populations under consideration or the ratio of the time units of the two samples In many situations the alternatives to (4) which are of interest will be one-sided. For example, after a factory has instituted a safety campaign, it would be of interest to see if the rate was unaffected as against the possibility of the rate having decreased; hence the alternatives to (4) would be

$$\frac{m_y}{m_x} < r. \tag{5}$$

In terms of the parameters introduced in (2), the hypothesis (4) and its alternatives (5) become

$$p = \frac{1}{1+r} \quad \text{and} \quad p > \frac{1}{1+r}. \tag{6}$$

Consider the probability given by (3) in much the same manner as was done in [5] This probability depends upon two parameters, μ and p, only the latter of which is specified by the hypothesis; consequently if critical regions independent of μ are desired, it will be necessary to find similar regions [6] with respect to μ. Since x and y are discrete variables, it is not possible to find similar regions of arbitrary size, consequently it will be necessary to introduce continuous approximating functions if such regions are desired and if best critical regions are to be found Toward this end consider the expression for $P(x, y)$ in (3) It states that the probability that x and y will take on specified values is the Poisson probability that the sample point will fall on the line $x + y = n$, multiplied by the binomial conditional probability that the point will have the specified x coordinate when the point is known to lie on this line. If p and n are not small, this binomial function could be approximated well by means of a normal function. Or, if desired, factorials could be replaced by corresponding gamma functions and the necessary normalizing factor introduced. Regardless of what continuous function is chosen, a region on each line $x + y = n$ ($n = 0, 1, 2, \cdots$) can be selected such that the conditional probability for this approximating function is α that a point on that line will lie in that region Most natural approximating functions would become trivial for $n = 0$; therefore it may be

necessary to choose an artificial function for this case or to adopt a convention of letting the origin be the critical region for this case but accepting only 100α percent of samples for which $n = 0$ as belonging to this critical region. The totality of such α regions will constitute a critical region of size α which is independent of μ because from (3) the probability of a point lying in this critical region would now be given by

$$\sum_{n=0}^{\infty} \frac{e^{-\mu}\mu^n}{n!} \cdot \alpha = \alpha \sum_{n=0}^{\infty} \frac{e^{-\mu}\mu^n}{n!} = \alpha.$$

Thus, similar regions with respect to μ of size α can be obtained by selecting regions of size α on each line $x + y = n$.

The preceding method for obtaining similar regions is the only method for doing so if such regions are restricted to be found on the lines $x + y = n$, because if a region of size α_n were selected on each line $x + y = n$, it would be necessary that

$$\sum_{n=0}^{\infty} \frac{e^{-\mu}\mu^n}{n!} \cdot \alpha_n = \alpha$$

independent of μ. This is equivalent to requiring that

$$e^{\mu} = \sum_{n=0}^{\infty} \frac{\alpha_n}{\alpha} \frac{\mu^n}{n!},$$

but since the power series for e^{μ} is unique, it follows that $\alpha_n = \alpha$.

3. Common best critical region. Among these similar regions there will exist a best critical region for testing the hypothesis $p = p_0$ against the single alternative $p = p_1$ if there exist best critical regions on each line $x + y = n$. From (6) it will be observed that this formulation is equivalent to testing the hypothesis $r = r_0$ against the single alternative $r = r_1$. The best critical region [6] on such a line, if it exists, will be that region which satisfies the inequality

$$\frac{f(x;\, p_0)}{f(x;\, p_1)} \leq k, \tag{7}$$

where f denotes the continuous function selected to approximate the binomial distribution on this line and k is a constant determined so that the probability, under the hypothesis $p = p_0$, will be α that a point on this line will lie in this region. If the normal approximating function with $m = np$ and $\sigma^2 = npq$ is used, (7) becomes

$$\sqrt{\frac{p_1 q_1}{p_0 q_0}}\, e^{\frac{1}{2}\left[\frac{(x-np_1)^2}{np_1q_1} - \frac{(x-np_0)^2}{np_0q_0}\right]} \leq k. \tag{8}$$

After completing the square in x, it will be found that this inequality reduces to

$$e^{\frac{1}{n}(1/p_1q_1 - 1/p_0q_0)\left[x - \frac{n(1/q_1 - 1/q_0)}{1/p_1q_1 - 1/p_0q_0}\right]^2} \leq c, \tag{9}$$

where c is independent of x.

If x_0 is a value of x such that

$$P[x > x_0 \mid p = p_0] = \alpha, \tag{10}$$

then (9) will hold for $x > x_0$ provided that $p_1 > p_0$. To demonstrate this fact, it is convenient to consider the three cases $p_0 + p_1 \gtrless 1$ separately. If $p_0 + p_1 > 1$,

$$\frac{1}{q_1} - \frac{1}{q_0} > 0, \qquad \frac{1}{p_1 q_1} - \frac{1}{p_0 q_0} > 0, \qquad \frac{1}{q_1} - \frac{1}{q_0} > \frac{1}{p_1 q_1} - \frac{1}{p_0 q_0},$$

and therefore $x \leq n \leq n\left(\frac{1}{q_1} - \frac{1}{q_0}\right) \Big/ \left(\frac{1}{p_1 q_1} - \frac{1}{p_0 q_0}\right)$. Since the coefficient of the brackets in (9) which involves x is positive, increasing x will reduce the left side of (9). If $p_0 + p_1 < 1$,

$$\frac{1}{p_1 q_1} - \frac{1}{p_0 q_0} < 0$$

and

$$\frac{n(1/q_1 - 1/q_0)}{1/p_1 q_1 - 1/p_0 q_0} < 0.$$

Since the coefficient is now negative, increasing x will reduce the left side of (9). Finally, if $p_0 + p_1 = 1$, (9) will reduce to

$$e^{\left[\frac{1}{p_1} - \frac{1}{p_0}\right]\left[x - \frac{n}{2}\right]} \leq k.$$

Since $1/p_1 - 1/p_0 < 0$, increasing x will decrease the left side of this inequality. It therefore follows that the region defined by (10) is a best critical region for every alternative of the form $p_1 > p_0$ on the line $x + y = n$. The totality of such regions for $n > 0$, together with the previously mentioned convention for $n = 0$, then constitutes a common best critical region among all possible similar regions for testing the hypothesis (4) against the set of alternatives (5).

In a similar manner it will be found that if the inequality in (10) is reversed, the critical region so defined, together with the convention, will constitute a common best critical region for every alternative of the form $p_1 < p_0$. If the alternative hypotheses consist of $p \neq p_0$, there will not exist a common best critical region using these approximating functions.

The critical region proposed in [5] is that for the special hypothesis $p_0 = \frac{1}{2}$ and the set of alternatives $p \neq p_0$. It will be found that the lower half of this critical region for $P = 2\alpha$ will differ little, except for very small samples, from that given by (10) for this special case; however, it possesses the disadvantage of being numerical and therefore of requiring a special table. The critical region given by (10) does not possess this disadvantage. This fact will be demonstrated in the next section.

4. Chi-square test. Consider the problem of testing compatibility between observed and expected frequencies in two cells. Let x and y represent the ob-

served frequencies and e_x and e_y the expected frequencies in a sample of size n. If the probability that an observation will fall in the first cell is, as in (6), $p = \frac{1}{1+r}$, then

$$e_1 = np = \frac{x+y}{1+r}$$

and

$$e_2 = n(1-p) = \frac{r(x+y)}{1+r}.$$

The chi-square function for testing compatibility then reduces to

$$\chi^2 = \sum_{i=1}^{2} \frac{(o_i - e_i)^2}{e_i} = \frac{(y-rx)^2}{r(y+x)}. \tag{11}$$

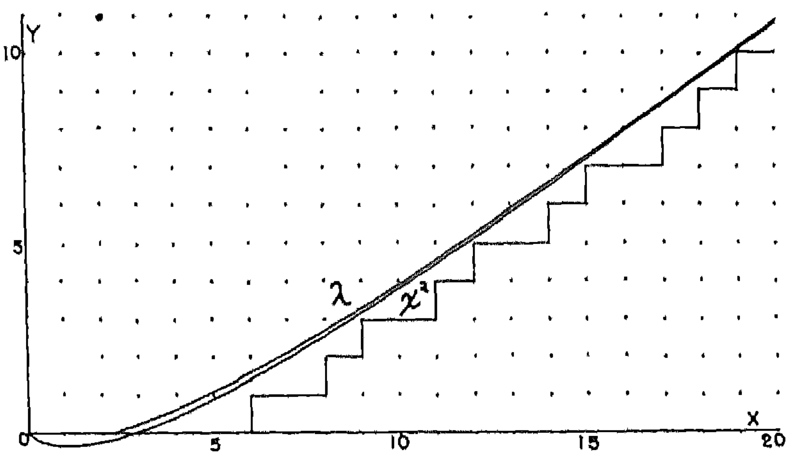

FIGURE 1.

Let χ_0^2 be the value of χ^2 such that $P[\chi^2 > \chi_0^2] = 2\alpha$ for one degree of freedom. With χ^2 replaced by χ_0^2 in (11), this equation determines a parabola in the x, y plane. If $x + y = n$ is not small, the probability of a point on the line $x + y = n$ lying outside of this parabola will be approximately 2α, the accuracy depending on the accuracy of the χ^2 approximation, and hence the probability of a point lying outside of and below this parabola will be approximately α. Thus, a critical region for testing $p = p_0$ against $p > p_0$ will be given by that part of the positive x, y plane which lies below this parabola. In Figure 1 the lower half of this parabola for the special case of $p_0 = \frac{1}{2}$ is indicated by the symbol χ^2. The critical region for the alternatives $p < p_0$ would be the region lying above the upper half of this same parabola, while the critical region for the alternatives $p \neq p_0$ would consist of both of these regions at the 2α level. For one degree of freedom, χ has a standard normal distribution; consequently the critical region given by (11) is the same as that given by (10) in which a normal approximation is used

on each line $x + y = n$ This equivalence is easily verified by replacing y by $n - x$ and r by q/p in (11)

5. Likelihood ratio test. The chi-square test of the preceding section yields a common best critical region for testing (4) against (5) for the normal approximation. It is interesting to compare this critical region with that obtained by the maximum likelihood principle, which requires no such approximations. Consider, therefore, the two dimensional parameter space

$$\Omega: \qquad m_x > 0, \qquad m_y > 0,$$

and the subspace

$$\omega \cdot \qquad \frac{m_y}{m_x} = r.$$

Maximizing P in (1) over Ω yields $\hat{m}_x = x$ and $\hat{m}_y = y$. Maximizing P over ω, treating P as a function of m_x, yields $\hat{m}_x = x + y/1 + r$. Then the maximum likelihood ratio becomes

$$\lambda = \frac{\max P\omega}{\max P_\Omega} = \frac{e^{-(x+y)}\left(\dfrac{x+y}{1+r}\right)^{x+y} r^y}{x!\,y!} \div \frac{e^{-(x+y)}\,x^x\,y^y}{x!\,y!}.$$

This reduces to

$$\lambda = \left(\frac{x+y}{1+r}\right)^{x+y} \cdot \frac{r^y}{x^x\,y^y}. \tag{12}$$

For a fixed value of λ, this equation determines a curve in the x, y plane which may be used to determine a critical region Since $-2 \log \lambda$ is known to possess an asymptotic chi-square distribution under certain conditions [7], choose as critical region that part of the positive x, y plane lying below the curve determined by (12) when λ has been replaced by λ_0, where λ_0 is determined from $-2 \log \lambda_0 = \chi_0^2$. This curve may be plotted by reducing it to the parametric form

$$x = \frac{\log \lambda_0}{(1+v)\log\dfrac{1+v}{1+r} + v \log\dfrac{r}{v}}, \qquad y = vx$$

A comparison of the critical regions corresponding to (11), (12), and a slight modification of [5] for the special case of $p_0 = \frac{1}{2}$ and $\alpha =$ 05 is given in the accompanying sketch. The modification of [5] consists in choosing x_0 to be that integer which most nearly satisfies (10), rather than to be the smallest integer for which the left side of (10) does not exceed α The latter method of choosing x_0 has a tendency to make the first type of error considerably smaller than α for small values of n. It will be observed that there are no appreciable differences between the maximum likelihood and chi-square critical regions Furthermore, it will be found that there are only two values of n, namely $n = 3$ and $n = 9$, for $n \leq 30$

for which the chi-square test and the modification of [5] might yield different decisions at this significance level.

The preceding sections show that the chi-square test is highly satisfactory for testing the homogeneity of two Poisson frequencies, except possibly for very small frequencies, and that therefore special numerical tables are not necessary.

6. Several Poisson frequencies. The generalization of (11) for a set of k frequencies is, of course, the ordinary chi-square function

$$\chi^2 = \sum_{i=1}^{k} \frac{(x_i - np_i)^2}{np_i}, \tag{13}$$

where $n = \sum_{i=1}^{k} x_i$, p_i is proportional to the sampling unit from which x_i was obtained, and $\sum_{i=1}^{k} p_i = 1$. The Poisson index of dispersion is merely a special case of (13) when $p_i = 1/k$. The adequacy of (13) for this special case has been studied elsewhere [3], [8], while studies of (13) in general are numerous and well known.

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SOME COMBINATORIAL FORMULAS ON MATHEMATICAL EXPECTATION

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The main problem considered here may be stated as follows:

Let $f_1(x), \cdots, f_n(x)$ be n polynomials. It is the purpose of this paper to establish formulas concerning the mathematical expectation (probable value) of the product

$$f_1(x_1) \cdots f_n(x_n),$$

where $x_1, \cdots, x_n$ are positive random variables and the sum of these is supposed known.

Before establishing the formulas let us introduce some notations for convenience.

1. Notation. (A) In this paper the notation $(m; k; x_1, \cdots, x_n)$ or $(m; k, x)$ is used to denote that a set of numbers $(x_1, \cdots, x_n)$ is over all different compositions of m into n parts with each $x \geqq k$, i.e. over all different integer solutions of the equation $x_1 + \cdots + x_n = m$ with each $x \geqq k$

(B) Let m, δ be two positive real numbers. The notation $E(m, \delta, [f_1] \cdots [f_n])$ denotes the mathematical expectation of the product $f_1(x_1) \cdots f_n(x_n)$ in which the sum $m = x_1 + \cdots + x_n$ is known and for every $x_\nu (\nu = 1, \cdots, n)$ the value of x_ν/δ is a positive integer. The notation $E(m, \delta, [f_1] \cdots [f_n])$ thus implies that the value of m is a multiple of δ. We call the δ a "varying unit", i.e. the least possible difference between two different quantities x_i and x_j, $i \neq j$. The notation $E(m\delta, [f]^n)$ is merely a special case that denotes the mathematical expectation of the product $f_1(x_1) \cdots f_n(x_n)$ under the known conditions

$$f_1 = \cdots f_n = f, \qquad x_1 + \cdots + x_n = m, \qquad \frac{x_\nu}{\delta} = \left[\frac{x_\nu}{\delta}\right] \geqq 1, \qquad (\nu = 1, \cdots, n),$$

where [ ] represents "integral part of".

(C) In order to simplify our formulas we always denote $f(x)$ by $f^{(x)}$, $f_{\nu_1} + \cdots + f_{\nu_s}$ by $f_{\nu_1 \cdots \nu_s}$ and $1.p_1 + \cdots + k.p_k$ by $\sigma(p)$ or σ. It is a convention that $\binom{m}{n} = 0$ for $m < n$.

2. Lemmas. Lemma 1. *Let $m, r_1, \cdots, r_n$ be non-negative integers. Then*

$$\sum_{(m;0;x)} \prod_{\nu=1}^{n} \binom{x_\nu}{r_\nu} = \binom{m+n-1}{r_1 + \cdots + r_n + n - 1}. \tag{1}$$

PROOF: The lemma follows immediately by considering the coefficient of the term $x^{m-(r_1+\cdots+r_n)}$ on both sides of

$$\left(\frac{1}{1-x}\right)^{r_1+1}\cdots\left(\frac{1}{1-x}\right)^{r_n+1}=\left(\frac{1}{1-x}\right)^{r_1+\cdots+r_n+n}$$

LEMMA 2. *Let a, b, c, $\cdots$ be any constants, and k_1, k_2, k_3, $\cdots$ any positive integers. Then*

$$\tag{2} \begin{aligned}&\sum_{(m,1;x)}\prod_{\nu=1}^{n}\left[a\binom{x_\nu}{k_1}+b\binom{x_\nu}{k_2}+c\binom{x_\nu}{k_3}+\cdots\right]\\ &\quad=n!\sum_{(n,0,\alpha,\beta,\gamma,\ )}\binom{m+n-1}{\alpha k_1+\beta k_2+\gamma k_3+\cdots+n-1}\frac{a^\alpha}{\alpha!}\frac{b^\beta}{\beta!}\frac{c^\gamma}{\gamma!}\cdots.\end{aligned}$$

PROOF: Expanding the left-hand side of (2) we see that the coefficient of the term $a^\alpha b^\beta c^\gamma\cdots$ is equal to

$$\frac{n!}{\alpha!\beta!\gamma!}\sum_{(m,0,x)}\binom{x_1}{k_1}\cdots\binom{x_\alpha}{k_1}\binom{x_{\alpha+1}}{k_2}\cdots\binom{x_{\alpha+\beta}}{k_2}\binom{x_{\alpha+\beta+1}}{k_3}\cdots\binom{x_{\alpha+\beta+\gamma}}{k_3}\cdots.$$

By Lemma 1 it becomes

$$\frac{n!}{\alpha!\beta!\gamma!}\binom{m+n-1}{\alpha k_1+\beta k_2+\gamma k_3+\cdots+n-1}.$$

Hence the lemma.

LEMMA 3. *Let m, $n(\leqq m)$ be two positive integers. Then, for any given polynomial $f(x)$ of the kth degree, we have*

$$\tag{3} \sum_{(m;1;x)}f(x_1)\cdots f(x_n)=n!\sum_{(n;0,p)}\binom{m+n-1}{\sigma+n-1}\prod_{\nu=0}^{k}\frac{[(f-1)^{(\nu)}]^{p_\nu}}{p_\nu!},$$

where $f^{(x)}=f(x)$, $\sigma=\sigma(p)=1.p_1+\cdots+kp_k$.

PROOF: Since $f(x)$ is a polynomial of the kth degree, there exist $(k+1)$ values $\beta_k,\cdots,\beta_0$ such that

$$\sum_{i=0}^{k}\beta_i\binom{x}{i}=f(x).$$

By putting $x=0,1,\cdots,k$, it is orderly determined that

$$\beta_\nu=f^{(\nu)}-\binom{\nu}{1}f^{(\nu-1)}+\cdots+(-1)^\nu\binom{\nu}{\nu}f^{(0)}=(f-1)^{(\nu)},\quad(\nu=0,1,\cdots,k).$$

The lemma is thus obtained by (2).

For convenience we denote the summation $\sum_{(m,1;x)}(m;1;x)\,f_1(x_1)\cdots f_n(x_n)$ by $S(m,[f_1]\cdots[f_n])$. Thus the formula (3) can be written as

$$S(m,[f]^n)=n!\sum_{(n;0;p)}\binom{m+n-1}{\sigma+n-1}\prod_{\nu=0}^{k}\frac{[(f-1)^{(\nu)}]^{p_\nu}}{p_\nu!}.$$

LEMMA 4. *Let $f_1(x), \cdots, f_n(x)$ be n given polynomials. Then*

$$(4)\qquad S(m, [f_1] \cdots [f_n]) = \frac{1}{n!} \sum_{\substack{(\nu_1 \cdots \nu_n) \\ 1 \leq k \leq n}} (-1)^{n-k} S(m, [f_{\nu_1} + \cdots + f_{\nu_k}]^n),$$

where $(\nu_1 \cdots \nu_k)$ runs over all different combinations out of $(1 \cdots n)$, $k = 1, \cdots, n$.

PROOF. The proof depends essentially on the formal logic theorem. Considering a typical term

$$\frac{n!}{q_1! \cdots q_t!} S(m, [f_{\nu_1}]^{q_1} \cdots [f_{\nu_t}]^{q_t}), \qquad 1 \leq t < n, \qquad q_1 + \cdots + q_t = n,$$

we see that it is contained in the last $(n - t + 1)$ summations of the righthand side of (4), i.e. in the summations $(\nu_1 \cdots \nu_k)$ as $k = t, t + 1, \cdots, n$. The number of occurrences of the term in the right-hand side of (4) is therefore

$$\sum_{\nu=0}^{n-t} (-1)^\nu \binom{n - t}{\nu} = \begin{matrix} 0 & \text{if} & t > n \\ 1 & \text{if} & t = n. \end{matrix}$$

The term vanishes generally except when $q_1 = \cdots = q_t = 1$. Hence the right-hand side gives

$$S(m, [f_1] \cdots [f_n])$$

3. Theorems with formulas. In the following statements of theorems and corollaries, the notation $(x_1 \cdots x_n)$ is always to denote a set of undetermined quantities, though the kind of the quantities of the set is stated.

THEOREM 1. *Let $(x_1 \cdots x_n)$ be a set of natural numbers under a known condition $x_1 + \cdots + x_n = m$. Then, for any given polynomial $f(x)$ of the kth degree, we have*

$$(5)\qquad E(m, 1, [f]^n) = \frac{n!}{\binom{m-1}{n-1}} \sum_{(n,0,p)} \binom{m + n - 1}{\sigma + n - 1} \prod_{\nu=0}^{k} \frac{[(f - 1)^\nu]^{p_\nu}}{p_\nu!}.$$

PROOF. Let $m' = m + nr$. By lemma 1 we then have

$$\sum_{(m,0,x)} \binom{x_1}{0} \cdots \binom{x_n}{0} = \sum_{(m',r,x)} 1 = \binom{m' - nr + n - 1}{n - 1}.$$

This is the number of compositions of m' into n parts with each part $\geqq r$. In particular, for $r = 1$ we see that the number of compositions of m into n parts is $\binom{m-1}{n-1}$. Thus by the definition of mathematical expectation, the required value is equal to

$$\frac{S(m, [f]^n)}{S(m, [1]^n)}, \qquad \text{i.e.} \qquad \binom{m - 1}{n - 1}^{-1} S(m, [f]^n)$$

The theorem is therefore proved by Lemma 3.

COROLLARY 1. *Let $(x_1 \cdots x_n)$ be a set of positive quantities, of which the varying unit is δ, and the sum is m. Then, for any given polynomial $f(x)$ of the kth degree, we have*

$$(6) \qquad E(m, \delta, [f]^n) = \frac{n!}{\binom{\frac{m}{\delta} - 1}{n - 1}} \sum_{(n,0,p)} \binom{\frac{m}{\delta} + n - 1}{\sigma + n - 1} \prod_{\nu=0}^{k} \frac{[(g - 1)^{(\nu)}]^{p_\nu}}{p_\nu!},$$

where

$$g(x) = f(\delta x), \qquad \sigma = 1p_1 + \cdots + kp_k.$$

PROOF: It is deduced by the relation $E(m, \delta, [f(x)]^n) = E(m/\delta, 1, [f(\delta x)^n)$.

COROLLARY 2. *Let $(x_1 \cdots x_n)$ be a set of non-negative real numbers under a known condition $x_1 + \cdots + x_n = m$. Then, for any given polynomial $f(x) = a_0 + \cdots + a_k x^k$, we have*

$$(7) \qquad E(m, 0, [f]^n) = \frac{(n!)^2}{n} \sum_{(n;0,q)} \frac{m^\sigma}{(\sigma + n - 1)!} \frac{(0!a_0)^{q_0}}{q_0!} \cdots \frac{(k!a_k)^{q_k}}{q_k!},$$

where

$$a_k \neq 0, \qquad \sigma = \sigma(q) = q_1 + \cdots + kq_k.$$

PROOF: The proof of the corollary depends essentially on the concept that two different real numbers may differ by an arbitrarily small number h.

Let h be an arbitrary positive number and let $f(xh) = h^k g(x, h)$, where the number k is the degree of $f(x)$. Then, since

$$\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} (n - \nu)^p = \begin{cases} 0 & \text{if } p > n \\ n! & \text{if } p = n \\ \binom{n+1}{2} n! & \text{if } p = n + 1, \end{cases}$$

we may write

$$\sum_{s=0}^{\nu} (-1)^s \binom{\nu}{s} g(\nu - s, h) = h^{\nu-k}[\nu! a_\nu + h \cdot R_\nu(h)],$$

$$\text{where} \quad \lim_{h\to 0} R_\nu(h) = \binom{\nu+1}{2} \nu! a_{\nu+1}.$$

Now we pass to the limit $h \to 0$, in which it is assumed that h runs through a sequence of rational numbers of the form $1/N$. Thus by Corollary 2 we have

$$\lim_{h\to 0} E(m, h, [f]^n) = n!(n - 1)! \sum_{(n;0;p)} \frac{m^\sigma}{(\sigma + n - 1)!} \prod_{\nu=0}^{k} \frac{(\nu! a_\nu)^{p_\nu}}{p_\nu!}.$$

Hence the corollary.

It may be noted that this corollary can also be independently deduced by the proportion of the two integrals:

$$\int \cdots \int_R f(x_1) \cdots f(x_n)\, dx_1 \cdots dx_{n-1}: \qquad \int \cdots \int_R dx_1 \cdots dx_{n-1},$$

where the integrals are all taken over the region R: $x_1 + \cdots + x_n = m$, $x_1 \geq 0, \quad x_n \geq 0$

Corollary 3 *Let $(x_1 \cdots x_n)$ be a set of positive real numbers under a known condition $a < x_1 + \cdots + x_n < b$, where a, b are non-negative numbers. Then, for any given polynomial $f(x) = a_0 + \cdots + a_k x^k$ $(a_k \neq 0)$, the mathematical expectation of the product $f(x_1) \cdots f(x_n)$, which we denote by $E((ab), 0, [f]^n)$, is given by the formula*

$$E(a, b), 0, [f]^n) = \frac{n!(n-1)!}{b-a} \times \sum_{(n,0,q)} \frac{b^{1+\sigma(q)} - a^{1+\sigma(q)}}{(1+\sigma(q))\cdot(n-1+\sigma(q))!} \frac{a_0^{q_0}}{q_0!} \cdots \frac{(k!a_k)^{q_k}}{q_k!}. \tag{8}$$

Proof: Since the required mathematical expectation is the mean

$$\frac{1}{b-a}\int_a^b E(u, 0, [f]^n)\, du,$$

Corollary 3 follows from Corollary 2.

On the other hand we see that

$$\lim_{h \to 0} E(a, a+h), 0, [f]^n) = E(a, 0, [f]^n).$$

Hence Corollary 2 can also be deduced from Corollary 3.

Theorem 2. (First generalization of Theorem 1). *Let $f_1(x), \cdots f_n(x)$ be n given polynomials, of which the highest degree is k. Then we have*

$$E(m, 1, [f_1] \cdots [f_n]) = \sum_{\substack{(\nu_1 \cdots \nu_s) \\ 1 \leq s \leq n}} \sum_{(n;0,p)} (-1)^{n-s} \times \frac{\binom{m+n-1}{\sigma+n-1}}{\binom{m-1}{n-1}} \prod_{\mu=0}^{k} \frac{[(f_{\nu_1 \cdots \nu_s} - 1)^{(\mu)}]^{p_\mu}}{p_\mu!}, \tag{9}$$

where

Proof: In the proof of theorem 1 we have seen that

$$E(m, 1, [f]^n) = \binom{m-1}{n-1}^{-1} S(m, [f]^n).$$

Thus, by similar reasoning and lemma 4, we have

$$E(m, 1, [f_1] \cdots [f_n]) = \sum_{\substack{(\nu_1 \cdots \nu_s) \\ 1 \leq s \leq n}} \frac{(-1)^{n-s}}{n!\binom{m-1}{n-1}} S(m, [f_{\nu_1 \cdots \nu_s}]^n).$$

The theorem is proved by lemma 3

COROLLARY 1. *Let δ be a varying unit. Then*

$$E(m, \delta, [f_1] \cdots [f_n]) = \sum_{\substack{(\nu_1 \cdots \nu_s) \\ 1 \leq s \leq n}} \sum_{(n,0,p)} (-1)^{n-s}$$

$$(10) \qquad \times \frac{\begin{pmatrix} \frac{m}{\delta} + n - 1 \\ \sigma + n - 1 \end{pmatrix}}{\begin{pmatrix} \frac{m}{\delta} - 1 \\ n - 1 \end{pmatrix}} \prod_{\mu=0}^{k} \frac{[(g_{\nu_1 \cdots \nu_s} - 1)^{(\mu)}]^{p_\mu}}{p_\mu !},$$

where

$$g_\nu(x) = f_\nu(\delta x), \qquad g_{\nu_1 \cdots \nu_s} = g_{\nu_1} + \cdots + g_{\nu_s}.$$

PROOF: By the relation $E(m, \delta, [f_1(x)] \cdots [f_n(x)]) = E(m/\delta, 1, [f_1(\delta x)] \cdots [f_n(\delta x)])$ we obtain the corollary.

COROLLARY 2. *For any positive real number m, we have*

$$(11) \qquad E(m, 0, [x^{p_1}] \cdots [x^{p_n}]) = \frac{p_1! \cdots p_n!(n-1)!}{(p_1 + \cdots + p_n + n - 1)!} m^{p_1 + \cdots + p_n}.$$

PROOF: Since $E(m, \delta, [f_1] \cdots [f_n]) = \sum (-1)^{n-s}/n! \, E(m, \delta[f_{\nu_1 \cdots \nu_s}]^n)$, we have, by letting $\delta \to 0$,

$$E(m, 0, [f_1] \cdots [f_n]) = \sum \frac{(-1)^{n-s}}{n!} E(m, 0, [f_{\nu_1 \cdots \nu_s}]^n).$$

The corollary is therefore deduced by (7).

THEOREM 3. (Second generalization of Theorem 1). *Let $(x_1 \cdots x_n)$ be a set of integers under known conditions $x_1 + \cdots + x_n = m$, $a \leq x_i \leq b$, where m, a, b are given integers. Then, for any given polynomial $f(x)$, the mathematical expectation of the product $f(x_1) \cdots f(x_n)$, denoted by $\underset{(ab)}{E}(m, 1, [f]^n$, is given by the formula*

$$(12) \qquad \underset{(a,b)}{E}(m, 1, [f]^n) = \frac{\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} S(m', [g]^\nu[h]^{n-\nu})}{\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} \binom{m'-1}{n-1}},$$

where

$g(x) = f(b + x)$, $h(x) = f(a + x - 1)$ *and* $m' = m - (a - 1)n + (a - b - 1)\nu$.

PROOF: Define $S(m, [f]^n) = 0$ for $m < n$, and $S(m, [f]^0) = \begin{matrix} 0 \text{ for } m > 0 \\ 1 \text{ for } m = 0 \end{matrix}$. We shall now prove that

$$\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} S(m', [g]^\nu[h]^{n-\nu}) = \sum_{\substack{(x_1 \cdots x_n) \\ a \leq x \leq b}} f(x_1) \cdots f(x_n),$$

where on the right-hand side of the expression the set $(x_1, \cdots x_n)$ under the summation runs over all different compositions of m into n parts and

$$a \leq x_\nu \leq b, \qquad \nu = 1, \cdots, n.$$

For convenience we denote the left-hand side of the expression by $\mathfrak{S}$, that is,

$$\mathfrak{S} = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} S(m', [g]^\nu [h]^{n-\nu})$$
$$= \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} \sum_{\bar{m}=\nu}^{m'} S(\bar{m}, [f(x+b)]^\nu) S(m' - \bar{m}, [f(x+a-1)]^{n-\nu}).$$

Let $f(\bar{x}_1) \cdots f(\bar{x}_n)$ be a product term contained in $\mathfrak{S}$, i.e., $\bar{x}_1 + \cdots + \bar{x}_n = m$; $\bar{x}_1 \geqq a, \cdots, \bar{x}_n \geqq a$. We assume that $\bar{x}_{\nu_1} \geq b + 1, \cdots, \bar{x}_{\nu_t} \geq b + 1$, where $\nu_1 \neq \nu_j$ if $i \neq j$. Then it is seen that the number of occurrences of the product term in $\mathfrak{S}$ is given by

$$\sum_{s=0}^{t} (-1)^s \binom{t}{s} = \begin{array}{ll} \text{if} & t \geq 1 \\ \text{if} & t = 0. \end{array}$$

Thus the product term $f(\bar{x}_1) \cdots f(\bar{x}_n)$ of $\mathfrak{S}$ vanishes except when

$$a \leq x_\nu \leq b, \qquad \nu = 1, \cdots, n.$$

Hence we have

$$\mathfrak{S} = \sum_{a \leq x \leq b} f(x_1) \cdots f(x_n).$$

Next, we shall find the number of different compositions of m into n parts with each $a \leq x_\nu \leq b$, i.e., the number of product terms of $\mathfrak{S}$. By the above result we see that the number is given by

$$\sum_{\nu=0}^{n} \sum_{m=\nu}^{m'} (-1)^\nu \binom{n}{\nu} \sum_{(\bar{m},1;x)} 1 \sum_{(m'-\bar{m};1;x)} 1 = \sum_{\nu=0}^{m} (-1)^\nu \binom{n}{\nu} \binom{m'-1}{n-1}.$$

Hence the theorem.

This theorem shows that the mathematical expectation $\underset{(ab)}{E}(m, 1, [f]^n)$ can be expressed by $S(\bar{m}[g]^\nu)$ and is therefore expressible in terms of linear combinations of the coefficients of the polynomial $f(x)$.

COROLLARY 1. *Let δ be a varying unit for which $\frac{m}{\delta}, \frac{a}{\delta}, \frac{b}{\delta}$ are all integers. Then*

$$\underset{(ab)}{E}(m, \delta, [f(x)]^n) = \underset{((a/\delta),(b/\delta))}{E}\left(\frac{m}{\delta}, 1, [f(\delta x)]^n\right).$$

COROLLARY 2. *Let $f_1(x), \cdots f_n(x)$ be n given polynomials. Then*

$$\underset{(ab)}{E}(m, 1, [f_1] \cdots [f_n]) = \sum_{\substack{(\nu_1 \quad \nu_s) \\ 1 \leq s \leq n}} \frac{(-1)^{n-s}}{n!} \underset{(a,b)}{E}(m, 1, [f_{\nu_1 \cdot \nu_s}]^n).$$

COROLLARY 3. *The number of integral solutions of the equation* $x_1 + \cdots + x_n = m$ *with* $a_1 \leq x_1 \leq b_1, \cdots, a_n \leq x_n \leq b_n$ *is equal to*

$$\sum_{\nu_1=0,\cdots,\nu_n=0}^{1,\cdots,1} (-1)^{\nu_1+\cdots+\nu_n} \cdot \binom{m+n-(a_1+\cdots+a_n)+(a_1-b_1-1)\nu_1+\cdots+(a_n-b_n-1)\nu_n-1}{n-1}.$$

PROOF: We have shown that the number of integral solutions of the equation $x_1 + \cdots + x_n = m$ with $a \leq x_\nu \leq b$ is given by

$$\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu}\binom{m-(a-1)n+(a-b-1)\nu-1}{n-1}.$$

Hence the number of integral solutions of the equation $x_{11} + \cdots + x_{1n_1} + \cdots + x_{s1} + \cdots + x_{sn_s} = m$ with $a_\nu \leq x_{\nu\mu} \leq b_\nu$, $(\nu = 1 \cdots s,\ \mu = 1, \cdots n_\nu)$, is given by

$$\sum_{\nu_1=0}^{n_1} \cdots \sum_{\nu_s=0}^{n_s} (-1)^{\nu_1+\cdots+\nu_s} \prod_{i=1}^{s} \binom{n_i}{\nu_i} \cdot \left[\sum_{(m,1,m_s)} \prod_{i=1}^{s} \binom{m_i-(a_i-1)n_i+(a_i-b_i-1)\nu_i-1}{n_i-1}\right]$$

$$= \sum_{\nu_1=0,\cdots,\nu_s=0}^{n_1,\cdots,n_s} (-1)^{\nu_1+\cdots+\nu_s} \binom{n_1}{\nu_1} \cdots \binom{n_s}{\nu_s} \cdot \binom{m-(a_1-1)n_1-\cdots-(a_s-1)n_s+(a_1-b_1-1)\nu_1+\cdots+(a_s-b_s-1)\nu_s-1}{n_1+\cdots+n_s-1}$$

The corollary follows at once by putting $n_1 = \cdots = n_s = 1$, $s = n$

This corollary can be restated in a more interesting manner as follows:

Let there be n store rooms, and let $b_1, \cdots, b_n$ be the numbers of stocks contained in 1st, 2nd, $\cdots$, n-th storerooms respectively. Then m stocks containing at least a_i stocks of the i-th storeroom $(i = 1, \cdots, n)$ can be chosen from these n storerooms in

$$\sum_{\nu_1=0,\cdots,\nu_n=0}^{1,\cdots,1} (-1)^{\nu_1+\cdots+\nu_n} \binom{m+n+(a_1-b_1-1)\nu_1+\cdots+(a_n-b_n-1)\nu_n-a_1-\cdots-a_n-1}{n-1}$$

different ways.

So far we have established several combinatorial formulas concerning the mathematical expectation of the product $f_1(x_1) \cdots f_n(x_n)$ under certain conditions. In the next section, we shall explain how to apply these formulas.

4. Applications. (a) *A criterion.* In order to make the above formulas applicable to practical problems we state a criterion as follows: The mathemati-

cal expectation of a function $F(x_1, \cdots, x_n)$ can be estimated by the above combinatorial formulas if and only if the sum of these undetermined quantities $x_1, \cdots, x_n$ is known and there exist n polynomials $f_1(x), \cdots, f_n(x)$ such that $F \propto f_1, \cdots, F \propto f_n$, where the quantities $x_1, \cdots, x_n$ may or may not be continuous. When the quantities are discontinuous, the varying unit is certainly given

(b) *Some approximations.* For $f(x) = \beta_0 + \cdots + \beta_k x^k (\beta_k \neq 0)$ we may write

$$(f - 1)^{(\nu)} = \sum_{s=0}^{k} \nu! \beta_s S_{\nu,s},$$

where $S_{\nu,s}$ is a Stirling number of the second kind, as used by Jordan, and defined by

$$\nu! S_{\nu,s} = \sum_{x=0}^{\nu} (-1)^{\nu-x} \binom{\nu}{x} x^s.$$

Thus, the formulas (5) and (9) can be written as follows:

$$(5') \quad E(m, 1, [f]^n) = \sum_{(n;0;p)} \frac{(m+n-1)!(m-n)!n!(n-1)!}{(m-\sigma)!(\sigma+n-1)!(m-1)!} \cdot \prod_{\nu=0}^{k} \frac{(\beta_\nu \bar{S}_{\nu,\nu} + \cdots + \beta_k \bar{S}_{\nu,k})^{p_\nu}}{p_\nu!}$$

$$(9') \quad E(m, 1, [f_1] \cdots [f_n]) = \sum_{\substack{(\nu_1 \cdots \nu_s) \\ 1 \le s \le n}} \sum_{(n,0;p)} (-1)^{n-s} \cdot \frac{(m+n-1)!(m-n)!n!(n-1)!}{(m-\sigma)!(\sigma+n-1)!(m-1)!} \prod_{\nu=0}^{k} \frac{(B_\nu \bar{S}_{\nu,\nu} + \cdots + B_k \bar{S}_{\nu k})^{p_\nu}}{p_\nu!},$$

where

$$\bar{S}_{\nu s} = \nu! S_{\nu s}, \qquad f_i = \beta_{i0} + \cdots + \beta_{ik} x^k, \qquad B_i = \beta_{1i} + \cdots + \beta_{ni}.$$

Now we state some convenient formulas concerning the number $\bar{S}_{\nu s}$.

If m is sufficiently large and t is smaller than m, the following recurrence relation is useful:

$$(13) \quad \begin{aligned} S_{m,m+t-1} &= \lambda_0 \binom{m+t-1}{t} + \lambda_1 \binom{m+t-1}{t+1} + \cdots + \lambda_{t-2} \binom{m+t-1}{2t-2} \\ S_{m,m+t} &= \binom{m+t}{t+1} + [(t+1)\lambda_0 + 2\lambda_1] \binom{m+t}{t+2} + \cdots + [(2t-1)\lambda_{t-2} + t\lambda_{t-1}] \binom{m+t}{2t}, \end{aligned}$$

where $\lambda_\nu \equiv 1$, $\lambda_{t-1} \equiv 0$ and $\lambda_1, \cdots, \lambda_{t-2}$ are all independent of m.

Starting from the first equality and using the recurrence relation $S_{m,n+1} = mS_{m,n} + S_{m-1,n}$ successively we have

$$\begin{aligned}
S_{m,m+t} &= \sum_{\nu=1}^{m} (m-\nu+1) S_{m-\nu+1,m+t-\nu} \\
&= \sum_{j=0}^{t-2} \lambda_j \left[\sum_{\nu=1}^{m} \binom{m+t-\nu}{t+j+1} (t+j+1) + \sum_{\nu=1}^{m} \binom{m+t-\nu}{t+j} (j+1) \right] \\
&= \sum_{j=0}^{t-2} \lambda_j \left[\binom{m+t}{t+j+2} (t+j+1) + \binom{m+t}{t+j+1} (j+1) \right] \\
&= \sum_{j=0}^{t-1} [(t+j)\lambda_{j-1} + (1+j)\lambda_j] \binom{m+t}{t+j+1},
\end{aligned}$$

where $\lambda_{-1} = \lambda_{t-1} = 0$. The recurrence relation is thus deduced.

Writing

$$S_{m,m+t} = \binom{m+t}{t+1} + \lambda_1 \binom{m+t}{t+2} + \cdots + \lambda_{t-1} \binom{m+t}{2t},$$

and using the recurrence relation as obtained above, the coefficients $\lambda_1, \cdots, \lambda_{t-1}$ may be exhibited as follows:

t	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
1								
2	3							
3	10	5						
4	25	105	105					
5	56	490	1260	945				
6	119	1918	9450	17325	10395			
7	246	6825	56980	190575	270270	135135		
8	501	22935	302995	1636635	4099095	4729725	2027025	
9	1012	74316	1487200	12122110	47507460	94594500	91891800	34459425

Now let

$$\bar{S}_{n,n+t} = \left[\binom{n+t}{t+1} + \lambda_1(t) \binom{n+t}{t+2} + \cdots + \lambda_{t-1}(t) \binom{n+t}{2t} \right] n!.$$

The recurrence relation obtained above gives

$$\lambda_{t-1}(t) = (2t-1)\lambda_{t-2}(t-1)$$

$$\lambda_{t-2}(t) = 2(t-1)\lambda_{t-3}(t-1) + (t-1)\lambda_{t-2}(t-1).$$

Thus we obtain

$$\lambda_{t-1}(t) = \frac{(2t)!}{t!\,2^t}.$$

$$\lambda_{t-2}(t) = (t-1)! \sum_{\nu=1}^{t-1} 2^{t-2\nu-1} \nu \cdot \binom{2\nu}{\nu}.$$

Let

$$\theta(t) = \sum_{x=1}^{t-1} \frac{x}{2^{2x}} \binom{2x}{x}.$$

Since the orders of $\binom{n+t}{t+1}, \ldots, \binom{n+t}{2t-1}$ are all less than $2t$ as $n \to \infty$, and since

$$\begin{aligned}\binom{n+t}{2t} \lambda_{t-1}(t) &= \frac{1}{t!}\left(\frac{n^2}{2}\right)^t \prod_{x=0}^{2t-1}\left(1+\frac{t-x}{n}\right)\\ &= \frac{1}{t!}\left(\frac{n^2}{2}\right)^t \left(1+\frac{t}{n}\right)\prod_{x=0}^{t-1}\left(1-\frac{x^2}{n^2}\right)\\ &= \frac{1}{t!}\left(\frac{n^2}{2}\right)^t\left(1+\frac{t}{n}\right)(1-O(n^{-2}))\\ &= \left(1+\frac{t-O(n^{-1})}{n}\right)\left(\frac{n^2}{2}\right)^t\frac{1}{t!},\end{aligned}$$

$$\begin{aligned}\binom{n+t}{2t-1}\lambda_{t-2}(t) &= \frac{2t}{n-t+1}\binom{n+t}{2t}(t-1)!\,2^{t-1}\theta(t)\\ &= \frac{4^t\theta(t)}{n-t+1}\binom{2t}{t}^{-1}\binom{n+t}{2t}\lambda_{t-1}(t)\\ &= \frac{4^t\theta(t)}{n-t+1}\binom{2t}{t}^{-1}\left(1+\frac{t-O(n^{-1})}{n}\right)\left(\frac{n^2}{2}\right)^t\frac{1}{t!}\\ &= \frac{4^t\theta(t)\binom{2t}{t}^{-1}+O(n^{-1})}{n}\left(\frac{n^2}{2}\right)^t\frac{1}{t!}.\end{aligned}$$

We may write (by Stirling's formula)

$$S_{n,n+t} = \left(\frac{n}{e}\right)^n\left(\frac{n^2}{2}\right)^t\frac{\sqrt{2\pi n}}{t!}\left(1+\frac{t+4^t\binom{2t}{t}^{-1}\theta(t)+\epsilon_n}{n}\right),$$

where $\epsilon_n \to 0$ as $n \to \infty$.

Now it is easily proved that the inequality

$$\sqrt{\frac{x}{\pi}} > \frac{x}{2^{2x}}\binom{2x}{x} > \sqrt{\frac{x-1}{\pi}}$$

holds for every positive integer x. We have, therefore,

$$\theta(t) < \sum_{x=1}^{t-1}\sqrt{\frac{x}{\pi}} < \int_1^t \sqrt{\frac{x}{\pi}}\,dx = \frac{2}{3\sqrt{\pi}}(t^{\frac{3}{2}}-1);$$

$$\theta(t) > \sum_{x=0}^{t-2}\sqrt{\frac{x}{\pi}} > \int_0^{t-2}\sqrt{\frac{x}{\pi}}\,dx = \frac{2}{3\sqrt{\pi}}(t-2)^{\frac{3}{2}};$$

and

$$\sqrt{t\pi} < 4^t \binom{2t}{t}^{-1} < \frac{t}{\sqrt{t-1}} \sqrt{\pi}.$$

Using these inequalities we have

$$l_t = \frac{2}{3} \sqrt{t}\,(t+2)^{\frac{1}{2}} < 4^t \binom{2t}{t}^{-1} \theta(t) < \frac{2}{3} \sqrt{\frac{t^2}{t-1}}\,(t^{\frac{1}{2}} - 1) = u_t ,$$

where it may be noted that

$$\lim_{t \to \infty} \frac{u_t}{l_t} = 1.$$

Hence we have in conclusion

$$(14) \quad \sum_{x=0}^{n} (-1)^{n-x} \binom{n}{x} x^{n+t} = \left(\frac{n}{e}\right)^n \left(\frac{n^2}{2}\right)^t \frac{\sqrt{2\pi n}}{t!} \left(1 + \frac{k(t) + t + \epsilon_n}{n}\right),$$

where

$$\frac{2}{3}\,(t-2)^{\frac{1}{2}} < \frac{k(t)}{\sqrt{t}} < \frac{2}{3} \sqrt{\frac{t}{t-1}}\,(t^{\frac{1}{2}} - 1).$$

Evidently the formula (14) implies (15) and (16):

$$(15) \quad \begin{aligned} &\sum_{x=0}^{n} (-1)^{n-x} \binom{n}{x} x^{n+t} \\ &\qquad \sim \left(\frac{n}{e}\right)^n \left(\frac{n^2}{2}\right)^t \frac{\sqrt{2\pi n}}{t!} \left(1 + \frac{2t^2}{3n}\right), \qquad t = 0(n^{\frac{1}{2}-\epsilon}), \qquad \epsilon > 0. \end{aligned}$$

$$(16) \qquad n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}. \qquad \text{(Stirling's formula).}$$

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ON THE CONSTITUENT ITEMS OF THE REDUCTION AND THE REMAINDER IN THE METHOD OF LEAST SQUARES

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1. Consider a set of variates y_i, $(i = 1, 2, \cdots, n)$, which are normally and independently distributed with variance 1. Let also a matrix (x_{ik}) with $i = 1, 2, \cdots, n; k = 1, 2, \cdots, s$ and rank s be given. Find $b_1, \cdots, b_s$ in terms of y_i so that

$$\psi^2 = \sum_i (y_i - \sum_k x_{ik} b_k)^2$$

is a minimum. This minimum value shall be denoted by $\psi^2_{\min}$.

It is known (see e.g. R. A Fisher, "Applications of Student's distribution", *Metron* Vol. 5, Part 3 (1925)) that $\psi^2_{\min}$ varies as does χ^2 with $n - s$ degrees of freedom and that it is possible to express $\psi^2_{\min}$ as the sum of $n - s$ squares of linear functions of the y_i. In the following lines $\sum_i y_i^2$ will be expressed as the sum of n squares of such functions which are independent and of variance 1. The sum of the first s squares will equal $\sum_i y_i^2 - \psi^2_{\min}$ and therefore the remaining $n - s$ squares equal $\psi^2_{\min}$.

Thus a simple way will be found of writing down explicitly the linear functions, whose existence only was proved by Professor Fisher in *Metron*.

2. We first calculate $\psi^2_{\min}$.

$\dfrac{\partial \psi^2}{\partial b_l} = 0$, for $l = 1, 2, \cdots, s$, gives the normal equations

$$\sum_{i=1}^{n} x_{il} y_i = \sum_{i=1}^{n} \sum_{k=1}^{s} x_{il} x_{ik} b_k, \tag{1}$$

which can be written

$$\sum_{i=1}^{n} x_{il} y_i = \sum_{k=1}^{s} X_{lk} b_k \tag{2}$$

with

$$X_{lk} = \sum_{i=1}^{n} x_{il} x_{ik}.$$

It follows from (1) that

$$\psi^2_{\min} = \sum_{i=1}^{n} y_i^2 - \sum_{i=1}^{n} \sum_{l=1}^{s} \sum_{k=1}^{s} x_{il} x_{ik} b_l b_k = \sum_{i=1}^{n} y_i^2 - \sum_{l=1}^{s} \sum_{k=1}^{s} X_{lk} b_l b_k, \tag{A}$$

where the b are solutions of (1).

3. A second expression for $\psi^2_{\min}$ can be found as follows: Introducing

$$c_i = \sum_{k=1}^{s} x_{ik} b_k$$

we obtain from (1)

$$\sum_{i=1}^{n} x_{il} c_i = \sum_{i=1}^{n} x_{il} y_i, \qquad (l = 1, 2, \cdots s). \tag{3}$$

Now if z_{iu}, $(u = s + 1, \cdots n)$, are any $n - s$ independent solutions of

$$\sum_{i=1}^{n} z_{iu} x_{il} = 0, \qquad (l = 1, 2, \cdots, s),$$

then the c_i satisfy also

$$\sum_{i=1}^{n} z_{iu} c_i = 0, \qquad (u = s + 1, \cdots n). \tag{4}$$

Let such a set of z_{iu} be chosen. Then (3) will be solved by

$$c_i = y_i - \sum_{v=s+1}^{n} \lambda_v z_{iv} \tag{5}$$

with λ_v as indefinite factors and these c_i satisfy (4), if

$$\sum_{i=1}^{n} z_{iu} y_i = \sum_{v=s+1}^{n} \sum_{i=1}^{n} z_{iu} z_{iv} \lambda_v, \qquad (u = s + 1, \cdots n), \quad \text{or} \quad \sum_{i=1}^{n} z_{iu} y_i = \sum_{v=s+1}^{n} Z_{uv} \lambda_v \tag{6}$$

with

$$Z_{uv} = \sum_{i=1}^{n} z_{iu} z_{iv} .$$

Because of (2) the equation (A) can be transformed into

$$\psi^2_{\min} = \sum_{i=1}^{n} y_i^2 - \sum_{l=1}^{s} \sum_{i=1}^{n} x_{il} y_i b_l = \sum_{i=1}^{n} y_i^2 - \sum_{i=1}^{n} y_i c_i = \sum_{i=1}^{n} \sum_{v=s+1}^{n} \lambda_v z_{iv} y_i$$

which is, because of (6)

$$\psi^2_{\min} = \sum_{u=s+1}^{n} \sum_{v=s+1}^{n} Z_{uv} \lambda_u \lambda_v , \tag{B}$$

where the λ are solutions of (6).

The comparison of (A) and (B) gives

$$\sum_{i=1}^{n} y_i^2 = \sum_{l=1}^{s} \sum_{k=1}^{s} X_{lk} b_l b_k + \sum_{u=s+1}^{n} \sum_{v=s+1}^{n} Z_{uv} \lambda_u \lambda_v$$

where the first form on the r h.s. shows the reduction of $\sum_{i=1}^{n} y_i^2$ by the method of least squares and the second form constitutes the remainder.

4. These two forms must now be expressed in terms of the y_i
We introduce the notations

$$X^{(1)} = X_{11}, \qquad X^{(2)} = \begin{vmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{vmatrix}, \quad \cdots \quad X^{(s)} = \begin{vmatrix} X_{11} & \cdot & X_{1s} \\ \cdot & & \cdot \\ X_{s1} & \cdots & X_{ss} \end{vmatrix}$$

and

$$Z^{(s+1)} = Z_{s+1,s+1}, \qquad Z^{(s+2)} = \begin{vmatrix} Z_{s+1\,s+1} & Z_{s+1\,s+2} \\ Z_{s+2\,s+1} & Z_{s+2\,s+2} \end{vmatrix} \quad \text{etc}$$

It is well known (and can easily be verified) that

$$\sum_{l=1}^{s} \sum_{k=1}^{s} X_{lk} b_l b_k = \frac{1}{X^{(1)}} (X_{11} b_1 + \cdots + X_{1s} b_s)^2$$
$$+ \frac{1}{X^{(1)} X^{(2)}} \left(\begin{vmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{vmatrix} b_2 + \cdots + \begin{vmatrix} X_{11} & X_{1s} \\ X_{21} & X_{2s} \end{vmatrix} b_s \right)^2$$
$$+ \cdots + \frac{1}{X^{(s-1)} X^{(s)}} X^{(s)^2} b_s^2$$

which may be written

$$\frac{1}{X^{(1)}} \left(\sum_{k=1}^{s} X_{1k} b_k \right)^2 + \frac{1}{X^{(1)} X^{(2)}} \begin{vmatrix} X_{11} & \sum_{k=1}^{s} X_{1k} b_k \\ X_{21} & \sum_{k=1}^{s} X_{2k} b_k \end{vmatrix}^2$$
$$+ \cdots + \frac{1}{X^{(s-1)} X^{(s)}} \begin{vmatrix} X_{11} & X_{12} & \cdots & \sum_{k=1}^{s} X_{1k} b_k \\ & & \cdots & \cdot \quad \cdot \\ X_{s1} & X_{s2} & & \sum_{k=1}^{s} X_{sk} b_k \end{vmatrix}^2$$

Using (2), this can be expressed in terms of the y_i instead of b_k as follows:

$$\frac{1}{X^{(1)}} \left(\sum_{i=1}^{n} x_{i1} y_i \right)^2 + \frac{1}{X^{(1)} X^{(2)}} \begin{vmatrix} X_{11} & \sum_{i=1}^{n} x_{i1} y_i \\ X_{21} & \sum_{i=1}^{n} x_{i2} y_i \end{vmatrix}^2$$

(7)

$$+ \cdots + \frac{1}{X^{(s+1)} X^{(s)}} \begin{vmatrix} X_{11} & X_{12} & \cdots & \sum_{i=1}^{n} x_{i1} y_i \\ & & \cdots & \cdot\cdot \\ X_{s1} & X_{s2} & \cdot & \sum_{i=1}^{n} x_{is} y_i \end{vmatrix}^2$$

Similarly by (6) the second form can be transformed into

$$(8)\quad \psi^2_{\min} = \frac{1}{Z^{(s+1)}}\left(\sum_{i=1}^{n} z_{i,s+1} y_i\right)^2 + \cdots + \frac{1}{Z^{(n-1)} Z^{(n)}} \begin{vmatrix} Z_{s+1,s+1} & Z_{s+1,s+2} & \cdots & \sum_{i=1}^{n} z_{i,s+1} y_i \\ \cdots & \cdots & \cdots & \cdots \\ Z_{n,s+1} & Z_{n,s+2} & \cdots & \sum_{i=1}^{n} z_{in} y_i \end{vmatrix}^2 .$$

The rank of (x_{ik}) is s, so that the order of the suffices can always be chosen so as to make the above denominators different from zero.

Thus both the reduction and the remainder have been expressed by sums of squares, whose numbers correspond to the "degrees of freedom" s and $n - s$ respectively.

5. It remains to be shown that the linear functions of the y_i appearing in each form are mutually orthogonal and that in every one of them the sums of the squares of the coefficients are unity.

Now if we call the n linear forms which occur above $\sum_{j=1}^{n} a_{ij} y_j$, $(i = 1, 2, \cdots, n)$, then our proof implies that

$$\sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} \left[\sum_{j=1}^{n} a_{ij} y_j\right]^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ij} a_{ik} y_j y_k .$$

This is an identity for any y_i, hence we must have

$$\sum_{i=1}^{n} a_{ij} a_{ik} = 1 \quad \text{if} \quad j = k, \quad \text{and}$$

$$= 0 \quad \text{if} \quad j \neq k.$$

We have thus shown that the matrix (a_{ij}) is orthogonal and it follows that

$$\sum_{i=1}^{n} a_{ji} a_{ki} = 1 \quad \text{if} \quad j = k \quad \text{and}$$

$$= 0 \quad \text{if} \quad j \neq k.$$

6. In practical applications the x_{ik} will be given and if the expression (7) or (8) is to be written down we must first solve the set of equations

$$\sum_{i=1}^{n} z_{iu} x_{il} = 0, \qquad (l = 1, 2, \cdots, s).$$

We may assume that

$$\begin{vmatrix} x_{11} & \cdots & x_{s1} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ x_{1s} & \cdots & x_{ss} \end{vmatrix} \neq 0.$$

There exist, of course, an infinity of solutions. A very simple one can be found if the matrix (x_{ik}) is completed into a square matrix by adding 1 in the diagonal places and 0 elsewhere We obtain

$$\begin{vmatrix} x_{11} & \cdots & x_{s1} & x_{s+1\,1} & \cdots & x_{ns} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{1s} & \cdot & x_{ss} & x_{s+1\,s} & \cdots & x_{ns} \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{vmatrix} \neq 0.$$

The minors of the terms of any of the $s + 1$th, $\cdots$ nth line give one of $n - s$ independent sets of solutions for the z_{iu}

If, e.g. $s = 1$, then the z_{iu} are

$$\begin{matrix} -x_{21} & x_{11} & 0 & 0 & \cdot \\ -x_{31} & 0 & x_{11} & 0 & \cdot \\ -x_{41} & 0 & 0 & x_{11} & \cdots \end{matrix}$$

etc.

and the Z are

$$\begin{matrix} x_{11}^2 + x_{21}^2, & x_{21}x_{31}, & x_{21}x_{41} \\ x_{21}x_{31}, & x_{11}^2 + x_{31}^2, & x_{31}x_{41} \\ x_{21}x_{41}, & x_{31}x_{41}, & x_{11}^2 + x_{41}^2 \end{matrix}$$

etc.

Hence, for $s = 1$, $n = 2$,

$$\psi_{\min}^2 = \sum_{i=1}^{n} y_i^2 - \frac{1}{x_{11}^2 + x_{21}^2} (x_{11}y_1 + x_{21}y_2)^2 = \frac{1}{x_{11}^2 + x_{21}^2} (-x_{21}y_1 + x_{11}y_2)^2$$

and for $s = 1$, $n = 3$

$$\psi_{\min}^2 = \sum_{i=1}^{n} y_i^2 - \frac{(x_{11}y_1 + x_{21}y_2 + x_{31}y_3)^2}{x_{11}^2 + x_{21}^2 + x_{31}^2}$$

$$= \frac{1}{x_{11}^2 + x_{21}^2} (-x_{21}y_1 + x_{11}y_2)^2 + \frac{\begin{vmatrix} x_{11}^2 + x_{21}^2 & -x_{21}y_1 + x_{11}y_2 \\ x_{21}x_{31} & -x_{31}y_1 + x_{11}y_3 \end{vmatrix}^2}{(x_{11}^2 + x_{21}^2)\begin{vmatrix} x_{11}^2 + x_{21}^2 & x_{21}x_{31} \\ x_{21}x_{31} & x_{11}^2 + x_{31}^2 \end{vmatrix}}.$$

If, however, $s = 2$, $n = 3$, then easy calculations lead to

$$\psi^2_{\min} = \sum_{i=1}^{n} y_i^{\,2} - \frac{(x_{11} y_1 + x_{21} y_2 + x_{31} y_3)^2}{x_{11}^2 + x_{21}^2 + x_{31}^2}$$

$$- \frac{\begin{vmatrix} x_{11}^2 + x_{21}^2 + x_{31}^2 & x_{11} y_1 + x_{21} y_2 + x_{31} y_3 \\ x_{11} x_{12} + x_{21} x_{22} + x_{31} x_{32} & x_{12} y_1 + x_{22} y_2 + x_{32} y_3 \end{vmatrix}}{(x_{11}^2 + x_{21}^2 + x_{31}^2) \begin{vmatrix} x_{11}^2 + x_{21}^2 + x_{31}^2 & x_{11} x_{12} + x_{21} x_{22} + x_{31} x_{32} \\ x_{12} x_{11} + x_{22} x_{21} + x_{32} x_{31} & x_{12}^2 + x_{22}^2 + x_{32}^2 \end{vmatrix}}$$

$$= \left(\begin{vmatrix} x_{21} & x_{31} \\ x_{22} & x_{32} \end{vmatrix} y_1 + \begin{vmatrix} x_{31} & x_{11} \\ x_{32} & x_{12} \end{vmatrix} y_2 + \begin{vmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{vmatrix} y_3 \right)^2$$

$$\div \left(\begin{vmatrix} x_{21} & x_{31} \\ x_{22} & x_{32} \end{vmatrix}^2 + \begin{vmatrix} x_{31} & x_{11} \\ x_{32} & x_{12} \end{vmatrix}^2 + \begin{vmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{vmatrix}^2 \right).$$

As a specialized case consider $s = 1$, and $x_{11} = x_{21} = \cdots = x_{n1} = 1$. Then the Z are

$$\begin{matrix} 2 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 1 & 1 & \cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 1 & \cdots & 2 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 2 \end{matrix}$$

and

$$\psi^2_{\min} = \sum_{i=1}^{m} y_i^2 = \frac{1}{n}\left(\sum_{i=1}^{n} y_i\right)^2 = \sum_{i=1}^{n} \left(y_i - \frac{\sum_{i=1}^{n} y_i}{n} \right)^2.$$

The sum of squares into which $\psi^2_{\min}$ can be transformed is then found to be

$$\frac{1}{2}(-y_1 + y_2)^2 + \frac{1}{2\cdot 3}(-y_1 - y_2 + 2y_3)^2$$

$$+ \frac{1}{3\cdot 4}(-y_1 - y_2 - y_3 + 3y_4)^2 + \cdots.$$[1]

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[1] This is the result contained in a paper by J. O. Irwin, "Independence of the constituent items in the analysis of variance" *Suppl. Roy. Stat. Soc. Jour.* Vol. 1 (1934).

NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

ON THE ANALYSIS OF A CERTAIN SIX-BY-SIX FOUR-GROUP LATTICE DESIGN USING THE RECOVERY OF INTER-BLOCK INFORMATION

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1. Introduction. A detailed description for a six-by-six four-group lattice design is given in a recent article [1] by the author, and the analysis is developed which uses only the intra-block information to correct the varieties for the block effects Here is developed the analysis that makes use of both the intra- and the inter-block information

Referring to Group X on page 307, [1], since block (1) contains varieties 1 to 6, and block (2) contains varieties 7 to 12, the difference between the means of these two blocks is also an estimate of the difference between the first six varieties and the second six varieties. The information obtained from such inter-block comparisons was ignored in the previous analysis. In attempting to use this information, the chief difficulty is to decide how estimates derived from the comparison of block totals shall be combined with the previous estimates. Since each block consists of six plots, comparisons between block totals may be expected to have a higher error variance than the within-block comparisons, just as in split-plot designs the main block comparisons usually have a higher error than the sub-plot comparisons. The problem is, therefore, to estimate the relative error variances of the inter- and intra-block comparisons, and then to combine the two types of estimates to the best advantage

2. Calculations of the adjusted varietal totals. In addition to the equations (7), [1], which contain all the intra-block information, we now have the additional set of equations,

$B_i = 6\mu +$ (sum varietal constants in this block) $+ \epsilon_i$, which are estimated by

$$B_i = 6m + \Sigma v_{bi} + E_i .$$

In these equations and all the following equations, the double prime symbol ($''$) used in [1] is omitted, but the statistics have the same meaning as in equations (7), [1] except in this paper they are adjusted by both inter- and intra-block information.

[1] The author wishes to express his appreciation to W G Cochran of Iowa State College, who advised in the preparation of this analysis.

The general problem is to minimize the function,

$$F = WS(y_{ij} - m - v_j - b_i)^2 + \frac{W'}{6}S(B_i - 6m - \Sigma v_{bi})^2$$

subject to the restriction $\sum_{j=1}^{36} v_j = 0$ and $\sum_{e=z}^{u} \sum_{i=1}^{k} b_{ei} = 0$, and where $W = \frac{1}{\sigma^2}$ and $W' = \frac{1}{\sigma_b^2}$.

Following the method given in [1] the typical block equations for $b_{z1} \cdots b_{z6}$ is

$$b_{z1} = \frac{1}{6}\frac{W}{3W + W'}(4B_{z1} - T_{z1}) = \frac{1}{6}\frac{W}{3W + W'}C_{z1}$$

and for $b_{t1} \cdots b_{u6}$ is

$$B_{t1} = \frac{1}{144}\left\{\frac{1}{(W + W')(3W + W')}[(25W^2 + 22\, WW' + W'^2)C_{t1} + (W - W')^2(C_{t3} + C_{t5})] + \frac{W - W'}{W + W'}(C_{u2} + C_{u4} + C_{u6})\right\}.$$

It can be seen that for $W' = 0$, b_{z1} and b_{t1} are the intra-block values given in [1] and for $W' = W$ they are the randomized block values

A typical adjustment varietal total then becomes

$$4v_1 + 4m = V_1 - \frac{W - W'}{W}(b_{z1} + b_{y1} + b_{t1} + b_{u2}).$$

3. Estimation of W and W'. Following the method presented by Cochran [6] and Yates [3], the error of a block total may be written as

$$E_i = e_{i1} + e_{i2} + \cdots + e_{i6} + 6b_i^*$$

where

$$V(e) = \sigma^2 \text{ and } V(b^*) = \sigma_b^2.$$

Hence $V(E_i) = 6\sigma^2 + 36\sigma_b^2$ and component (a) is thus an estimate of $\sigma^2 + 6\sigma_b^2$. One finds from evaluating the expected value of (15), [1] corrected for replicates, $E\left(\Sigma bC - \frac{\Sigma b \Sigma C}{6}\right)$, that the expected value of component (b) is $\sigma^2 + \frac{3}{4}\cdot 6\sigma_b^2$. In the analysis of variance if components (a) and (b) are pooled, one obtains the block variance B as an estimate of $\sigma^2 + \frac{7}{8}\cdot 6\sigma_b^2$. Since the intra-block variance is an estimate of σ^2 the estimates of the true variance between blocks, $\sigma^2 + 6\sigma_b^2$, is $\frac{8B - E}{7} = \frac{1}{W'}$.

4. Standard error of adjusted varietal means. The standard error of the difference between the adjusted means of two varieties which appear together in the same blocks in groups Z or U, is

$$\frac{1}{4kW}\left[(k - 2) + \frac{8W}{3W + W'}\right],$$

obtained by the method outlined by Cochran. Similarly, for the case in which the varieties are together in the same block in groups Z or U

When an attempt is made to express the difference between these two adjusted varieties which appear together in the same block in groups X or Y in terms of the levels of the main effects and interactions, the interactions are no longer unconfounded and the method employed above breaks down.

If one is willing to assume that the formula for the variance of the difference between two adjusted varietal means for varieties which appear together in the same block in the groups X or Y is of the form $\frac{1}{24\,W}\left(A + \frac{BW}{3W + W'}\right)$ the constants may be determined by the values already known, [1] This form can be shown to be that for a quadruple lattice.

The formula $\frac{1}{24W}\left(A + \frac{BW}{3W + W'}\right)$ must reduce to the value for intra-block analysis [1] when $W' = 0$, and when $W = W'$ to the value for complete randomized blocks When these conditions are imposed, the formula becomes

$$\frac{1}{144W}\left(16 + \frac{80W}{3W + W'}\right).$$

This value is slightly larger than the value obtained when the adjusted varieties appear together in the same block in groups Z or U, as should be the case. This gives us a lower limit. One can arrive at the upper limit in the following manner: suppose the variance $(\text{intra})_1$ obtained in the intra-block analysis for the difference between two varietal means such as v_1 and v_2 is greater than that for varietal means v_8 and v_1 $(\text{intra})_2$, then it follows that:

$$(\text{inter} + \text{intra})_1 \leqq (\text{inter} + \text{intra})_2 \times \frac{(\text{intra})_1}{(\text{intra})_2}$$

Using this relation, the upper limit for two varieties together in the same block in groups X or Y is

$$\frac{1}{24W}\left(3 + \frac{12W}{3W + W'}\right)\frac{64}{63},$$

which gives a value slightly greater than the formula derived, as it should if it is to be the upper limit. In a similar manner one gets the variance for the difference between varietal means not appearing together in the same block.

5. Efficiency of the design to the randomized complete blocks. By the method outlined by Cochran [6] the efficiency can be shown to be measured by the ratio of

$$\frac{\frac{k}{W} + \frac{1}{W'}}{k + 1}$$ to 4 (average error variance of the difference between two plots).

It will be noted, by using the above formula, that the gain in efficiency for the numerical problem given in [1] is 1.003, which for our purpose here is zero.

This, in general, will not be the case, for on most soils there is a block difference. In this particular test the ground used had been previously filled in with well mixed soil. The efficiency for the analysis given in [1] relative to the randomized complete blocks was less than 1.00

This paper and the previous one show what a long tedious procedure is necessary to analyze the data, when the design does not follow the rules for the construction of the lattice, triple lattice, etc. The complexity of these methods stresses the importance, to those designing experiments, of not deviating from the established design if the most information is to be secured from the data with simple calculations.

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FURTHER REMARKS ON LINKAGE THEORY IN MENDELIAN HEREDITY

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In the following an explicit formula for the distribution of genotypes in case of three Mendelian characters will be given [formula (5)]. The complete discussion of the case $m = 3$ suggests a supplement (as stated in the last paragraph of this paper) to the general limit theorem dealing with m characters.

In an earlier paper[1] recurrence formulae have been derived which furnish the distribution of genotypes in the nth generation if the distribution in the $(n - 1)$th generation and the "linkage distribution" (l.d.) are known. It was also shown how to "integrate" this system of difference equations so as to determine the distribution in the nth generation directly from that in the 0th generation. This last method, though straightforward, requires however in each particular case quite a few operations.

In case m, the number of Mendelian characters, equals two, an explicit formula for the problem in question had been known. Denote by $p(x_1, x_2)$,

[1] HILDA GEIRINGER, *Annals of Math. Stat.* Vol. 15 (1944), pp.25-57. The notation in the present Note will be the same as in this paper.

($x_1, x_2 = 1, 2, \cdots k$), the "distribution of transmitted genes" in the original, 0th, generation, by $p^{(n)}(x_1, x_2)$ that in the nth generation and by c the "crossover probability" (c.p.). Then the simple formula holds:[2]

$$(1)\qquad p^{(n)}(x_1, x_2) = (1 - c)^n p(x_1, x_2) + [1 - (1 - c)^n] p_1(x_1) p_2(x_2).$$

This may also be written:

$$(1')\qquad p^{(n)}(x_1, x_2) = p_1(x_1)p_2(x_2) + (1 - c)^n[p(x_1, x_2) - p_1(x_1)p_2(x_2)],$$

where $p_i(x_i)$ are the marginal distributions derived from $p(x_1, x_2)$. (1′) shows that, if in case of independence of the original distribution, $p(x_1, x_2) = p_1(x_1)p_2(x_2)$ then $p^{(n)}(x_1, x_2) = p(x_1, x_2)$ for every n. The same is true for arbitrary $p(x_1, x_2)$ if $c = 0$. Otherwise, if $c > 0$ the second term to the right in (1′) tends towards zero as $n \to \infty$ and the well known limit theorem results.

In case $m = 3$, a remarkably elegant explicit formula exists[3] which may be deduced from the author's general theory. In this case the l d. is completely equivalent to the three c.p.'s c_{12}, c_{23}, c_{13}. The c_{ij} are probabilities with sum $\leqq$ 2, and for which the triangular relation

$$(2)\qquad c_{ij} + c_{jk} \geqq c_{ik}$$

holds If $l(\epsilon_1, \epsilon_2, \epsilon_3)$ ($\epsilon_i = 0, 1$) denotes the eight values of the l.d. we have (see quot. [1], p 32) $l(000) = l(111), l(100) = l(011), l(010) = l(101), l(001) = l(110)$, hence three independent values only. We may introduce

$$(3)\qquad \begin{aligned} 2l(000) = v(000) = v_0,\ 2l(100) &= v(100) \equiv v_1,\ 2l(010) = v(010) \equiv v_2 \\ 2l(001) &= v(001) \equiv v_3;\quad v_0 + v_1 + v_2 + v_3 = 1. \end{aligned}$$

It follows easily that

$$(4)\qquad c_{ij} = v_i + v_j, \qquad (i \neq j,\ i, j = 1, 2, 3).$$

The original distribution $p(x_1, x_2, x_3)$ has marginal distributions $p_{ij}(x_i, x_j)$, $p_i(x_i)$. These values will be denoted briefly by $p_{123}, p_{12}, p_{23}, p_{13}, p_1, p_2, p_3$ respectively. Writing in an analogous way $p^{(n)}(x_1x_2x_3) = p_{123}^{(n)}$ the new formula is the following:

$$(5)\qquad \begin{aligned} p_{123}^{(n)} = p_1p_2p_3 &+ [(v_0 + v_1)^n - v_0^n](p_1p_{23} - p_1p_2p_3) + [(v_0 + v_2)^n - v_0^n](p_2p_{13} \\ &- p_1p_2p_3) + [(v_0 + v_3)^n - v_0^n](p_3p_{12} - p_1p_2p_3) + v_0^n(p_{123} - p_1p_2p_3). \end{aligned}$$

This useful formula permits to compute readily $p_{123}^{(n)}$ for every n. In terms of the c_{ij}, writing

$$(6)\qquad d_{ij} = 1 - c_{ij}, \qquad v_0 = 1 - \tfrac{1}{2}(c_{12} + c_{23} + c_{13}),$$

it reads

$$(5')\qquad p_{123}^{(n)} = p_1p_2p_3 + (d_{23}^n - v_0^n)(p_1p_{23} - p_1p_2p_3) + \cdot + \cdot\ v_0^n(p_{123} - p_1p_2p_3)$$

[2] H. S. Jennings, *Genetics*, Vol. 12 (1917) pp 97–154.

[3] Professor Felix Bernstein called this author's attention to the biologically interesting case $m = 3$

In these formulae the rôle of independence of the original distribution is clearly seen: If $p_{ij} = p_i p_j$ and $p_{123} = p_1 p_2 p_3$ then $p_{123}^{(n)} = p_{123}$ for every n and every l.d. The same holds for every n and every p_{123} if $v_0 = 1$, which implies that all c_{ij} be zero. If in (5') all $d_{ij} < 1$, hence all $c_{ij} > 0$ the limit theorem $\lim_{n \to \infty} p_{123}^{(n)} = p_1 p_2 p_3$ results. $c_{ij} > 0$ means that complete linkage between any two genes is excluded. If, on the other hand, e.g. $v_0 > 0$, $v_1 > 0$, $v_0 + v_1 = d_{23} = 1$, $c_{23} = 0$, hence $v_0 < 1$, $v_2 = v_3 = 0$ we get $p_{123}^{(n)} \to p_1 p_{23}$. If $c_{23} = c_{13} = 0$ the triangular relation (2) shows that $c_{12} = 0$ too, a case considered above.

It should be noticed that (5) is, of course, in agreement with the author's equation (41) in quot. [1]. It only has to be observed,—an obvious fact not mentioned in my earlier paper,—that in the former setup the sum of all the $\alpha^{(n)}$ for every fixed m equals one. Thus for $m = 3$:

$$(7) \qquad \alpha_{123}^{(n)} + \alpha_{1,23}^{(n)} + \alpha_{2,13}^{(n)} + \alpha_{3,12}^{(n)} + \alpha_{1,2,3}^{(n)} = 1, \text{ (for every } n),$$

and

$$(8) \qquad \alpha_{123}^{(n)} = v_0^n, \qquad \begin{aligned} \alpha_{1,23}^{(n)} &= (v_0 + v_1)^n - v_0^n = d_{23}^n - v_0^n, \\ \alpha_{2,13}^{(n)} &= (v_0 + v_2)^n - v_0^n = d_{13}^n - v_0^n, \\ \alpha_{3,12}^{(n)} &= (v_0 + v_3)^n - v_0^n = d_{12}^n - v_0^n. \end{aligned}$$

The preceeding complete discussion of the case $m = 3$ suggests a remark concerning the general case of m characters. In my earlier paper the influence on the main limit theorem of certain ways of degeneration of the l.d. had not been explicitly considered. In the following we shall use the v-distribution which is a little shorter to write than the l.d. $l(\epsilon_1, \epsilon_2, \cdots \epsilon_m)$. The v-distribution contains only 2^{m-1} values with sum one, defined in a way similar to (3). The main limit theorem ([1], theorem II, p. 42) states in our present notation that

$$(9) \qquad \lim_{n \to \infty} p_{12\cdots m}^{(n)} = p_1 p_2 \cdots p_m,$$

if "complete linkage" between any group of genes is excluded. That implies that not only $v_0 \equiv v(0, 0, \cdots 0) = 1$ must be excluded but even $v_{i,\cdots k}(0, \cdots 0) = 1$, where this last probability denotes a marginal distribution of the v-distribution of an order $\geqq 2$. To assure this it is necessary and sufficient that no $v_{ij}(0, 0) = 1$, or no $d_{ij} \equiv v_{ij}(0, 0) = 1$, or no $c_{ij} = 0$. Hence (9) holds if and only if no $c_{ij} = 0$. If this condition is not satisfied the l.d. degenerates in various ways and the limit theorem is to be modified accordingly. If, in particular, $v_0 = 1$, *all* $c_{ij} = 0$, and $p_{12\cdots m}^{(n)} = p_{12\cdots m}$ for every n.

Between these two extreme cases ("no $c_{ij} = 0$", "all $c_{ij} = 0$") are the different possibilities of $r < m$ *groups of completely linked-characters* (see [1] p. 36, iv)). Consider e.g. $m = 7$ and $v_{1234}(0000) = 1$, $v_{567}(000) = 1$ (this is realized if $v(0000000) > 0$, $v(0000111) > 0$ with sum of these two numbers equal to one) then $\lim_{n \to \infty} p_{12\cdots 7}^{(n)} = p_{1234} p_{567}$. Here the four characters 1, 2, 3, 4 act as one character and $p_{1234}^{(n)} = p_{1234}$ for every n. Also $p_{567}^{(n)} = p_{567}$. Or if, for $m = 6$, $d_{12} = d_{34} = d_{56} = 1$ (realized if $v(000000) > 0$, $v(110000) > 0$, $v(001100) > 0$, $v(000011) > 0$, with

the sum of these four values equal to one) then $p_{12\cdots6}^{(n)} \to p_{12}p_{34}p_{56}$ If however for $m = 6$ merely $d_{12} = d_{34} = 1$ (realized if, in a notation analogous to (3), v_0, v_5, v_6, v_{56}, v_{12}, v_{34}, v_{125}, v_{126} are the only non-zero values of the l.d.) then $p_{12\cdots6}^{(n)} \to p_{12}p_{34}p_5p_6$.

In general, with a proof which consists in a modification of the reasoning (p. 41), of my earlier paper, we may state the following complement to the main limit theorem (9): *If the l.d. is such that $r < m$ disjoint groups G_1, G_2, $\cdots$ G_r of completely linked characters exist, i.e. such that within each group no crossover takes place, each group containing as many of the m numbers as compatible with the definition but not less than two, and all groups together containing $s \leq m$ of the m elements, then, as $n \to \infty$, $p_{12\cdots m}^{(n)}$ converges towards the product of those marginal distributions (of the original generation) which correspond to these groups multiplied by the marginal distributions of order one of the remaining free elements which are not contained in any such group In a formula*

$$\lim_{n\to\infty} p_{\sigma_1,\sigma_2,\cdots\sigma_r,\gamma_{s+1},\gamma_{s+2},\cdots\gamma_m} = p_{\sigma_1}p_{\sigma_2}\cdots p_{\sigma_r}p_{\gamma_{s+1}}p_{\gamma_{s+2}}\cdots p_{\gamma_m}. \tag{10}$$

We may also characterize these linked groups of maximum size by stating that while within each group no crossover takes place there must be at least one c.p. $\neq 0$ among any two such groups and at least one among any group and any free element. It may however be noted that if there is one c.p. > 0 among two groups of complete linkage (or among a group and a free element) then all c.p.'s among these two groups are different from zero. In fact, it follows by repeated use of the triangular relation (2) that if one c.p. among two disjoint groups of complete linkage is zero, all of them are zero. If, e.g., (1, 2, 3) and (5, 6, 8) are two groups of complete linkage, i.e. $v_{123}(000) = 1$ and $v_{568}(000) = 1$ and if besides $c_{15} = 0$, then $v_{123568}(000000) = 1$ and these six elements form a group of complete linkage.

It may be noticed that the above statement of the generalized limit theorem becomes simpler and more elegant by counting "free elements" as groups. It might then run as follows: *If G_1, G_2, $\cdots$ $G_t (t \leq m)$ are the maximal groups of completely linked characters, then, under the hypotheses of the earlier paper, the gene distribution in successive generations approaches a limit in which the original (marginal) probabilities within each group G_i are preserved and genes and sets of genes fromd ifferent groups are independently distributed.*

ON THE DEFINITION OF DISTANCE IN THE THEORY OF THE GENE

By Hilda Geiringer

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In several letters to this author Dr. I. M H. Etherington of the University of Edinburgh has raised questions concerning the author's definition of "distance" proposed in Section 10 of her paper on Mendelian heredity,[1] comparing it with

[1] *Annals of Math. Stat.*, Vol. 15 (1944), pp. 25–57

the definition implicit in Professor J. B. S. Haldane's earlier treatment.[2] The main content of the author's paper consists of some general limit theorems and the integration of a certain system of difference equations. The distance definition is a by-product subject to discussion.

"Distance" d_{ij} between two genes i and j is defined by the author as the *mathematical expectation of the number of crossovers in the interval* (i, j) with respect to the "linkage distribution" (l.d.). This basic concept is introduced as follows (page 32): If S is the set of numbers $1, 2, \cdots m$ (m being the number of Mendelian characters), A any subset of A and $A' = S - A$, we denote by $l(A)$ the probability that an individual with "maternal" genes $x_1, \cdots, x_m$ and paternal genes $y_1, \cdots, y_m$ transmit the paternal genes belonging to A and the maternal genes belonging to A'. These 2^m probabilities constitute the l.d. From these definitions the equality ((G. (53′))

$$d_{ij} = c_{i,i+1} + c_{i+1,i+2} + \cdots + c_{j-1,j} \qquad (i < j) \tag{1}$$

is derived, where c_{ij} is the probability of a "crossover" (c.p.) in (i, j). This distance has the required additivity: ((G. (54))

$$d_{ij} + d_{jk} = d_{ik}, \qquad (i < j < k). \tag{2}$$

Etherington points out that the term "distance" has an established currency in genetics being the basis on which chromosome maps are constructed, and that there is a standard method of calculating it in accordance with which (1) is an "approximation valid only when the adjacent c.p.'s are small." Moreover "the biological uniqueness has been lost for the value of d_{ij} now depends on the particular set of intermediate genes which we happen to be considering. If any of them are omitted from consideration then the inequality ((G. (13)).

$$c_{ij} + c_{jk} \geqq c_{ik} \tag{3}$$

shows that in general d_{ij} is diminished while if new genes are taken into consideration d_{ij} may increase." "In order that d_{ij} should not depend on a particular choice of intermediate genes the word 'crossover' in the definition given would have to be interpreted as 'chiasma' instead of 'odd number of chiasmata', and then d_{ij} cannot be evaluated in terms of the l.d. alone without further assumptions regarding the interference of crossovers."

The point of view adopted in the author's paper was to regard the *l.d.* as the basis from which everything else has to be inferred. The number m of Mendelian characters is considered constant and the distance, being a mathematical expectation with respect to the l.d. necessarily depends on it. In this conception distance is not a geometric property which can be measured for any two genes independently but rather a system of $m(m - 1)/2$ consistent numbers associated to the m genes. There is no choice regarding the intermediate genes to be taken into consideration; *all* known genes are to be considered, i.e. one has to use the available relevant information in order to determine the l.d., the c.p.'s and the

[2] Quotation [4a] in the author's paper. References to these papers will be distinguished by the initials H and G.

distances. If the information is incomplete the results will be provisional and subject to change; if it is satisfactory the same will be true for the distances. Thus it is nothing but natural that d_{ij} is changed if some genes are omitted from consideration, or if new genes are discovered. In this set up "crossover"—defined by means of the marginal distributions of second order of the l.d.—means a transition from the paternal to the maternal set or vice versa. (Expressed in terms of the chiasma-hypothesis this means "odd number of chiasmata between adjacent genes.") Additional assumptions "regarding the interference of crossovers" are neither necessary nor admissible. All this is contained in the l.d.

Haldane's approach as translated by Etherington into the author's notation is as follows. "The genes are considered to be distributed continuously along a chromosome. Thus this approach unlike G.'s is not based on the l.d. of a finite set of genes. We must think of one suffix, i, as referring to a gene at a fixed locus on the chromosome, the others to variable loci, so that the c.p.'s are variable. For any three genes i, j, k a quantity p is defined by the equation

$$c_{ik} = c_{ij} + c_{jk} - pc_{ij}c_{jk}, \qquad (i < j < k). \tag{4}$$

Biological considerations show that p is a number between 0 and 2 (small when c_{ij} and c_{jk} are both small, increasing, on the whole, with $c_{ij} + c_{jk}$). The distance D_{ij} is defined by the statement

$$D_{kj}/c_{kj} \to 1 \quad \text{as } k \text{ approaches } j \qquad (c_{kj} \to 0), \tag{5}$$

together with the additive property, and from this with (4) Haldane's general distance expression is derived:

$$D_{ij} = \int_0^{c_{ij}} \frac{dc_{ij}}{1 - p_0 c_{ij}}. \tag{6}$$

Here $p_0 \equiv p_0(c_{ij})$ denotes the limiting form of p when k approaches j, and represents biologically a property of the chromosome segment (i, j), a measure of interference. Any suitable specification of this function $p_0(c_{ij})$ would constitute a mathematical 'model' of the chromosome. If p were constant we should have $p_0 = p$ and

$$D_{ij} = -\frac{1}{p}\log\,(1 - pc_{ij}). \tag{7}$$

Both Haldane and Geiringer considered the special cases $p = 2$ (no interference) and $p = 0$ (complete interference) for which respectively

$$D_{ij} = -\tfrac{1}{2}\log\,(1 - \tfrac{1}{2}c_{ij}) \tag{7'}$$

$$D_{ij} = c_{ij} = d_{ij}. \tag{7''}$$

Since p is always between 0 and 2 Haldane concludes that the true value of D_{ij} is between (7′) and (7″), and he gives reasons for saying that (7′) is nearly correct for genes 'far apart,' (7″) for genes 'close together.' "

If the author is right, this seems to be the standard definition accepted in genetics as mentioned above by Etherington. A few, not exhaustive, comments may be added Writing in (6) t for the variable of integration and $p_0 = p_0(t)$ it is seen that the expression

$$D_{ij} = \int_0^{c_{ij}} \frac{dt}{1 - tp_0(t)} \tag{6}$$

contains the unknown function $p_0(t)$, which is unspecified except for the statement that it is bounded between 0 and 2. It is immediately seen that with an arbitrary $p_0(t)$ and without a restriction taking the place of (4) this distance (6) will not be additive in the sense of (2). By imposing, after a choice of $p_0(t)$, appropriate restrictions on the c_{ij} additivity may be achieved. For instance in the particular case $p_0(t) = p = \text{const}$, (2) holds by virtue of (4). For such a set of restrictions it has then to be proved that the corresponding "model" is "consistent," i e. that the so restricted c.p.'s form a compatible set of marginal distributions of second order of an m-variate distribution, the l.d.

These different points will be exemplified presently by studying the particular case $p_0(t) = p$, where p is a suitably chosen constant; the parameter p is to be fitted to the observations under consideration. It may be impossible to reproduce a set of observations satisfactorily if one parameter only is available. In fact, Haldane's paper suggests that it is not only the particular case $p = \text{const}$ he has in mind. It seems however that if D_{ij} is given by (6) with a non constant $p_0(t)$, complicated and perhaps (biologically) not very meaningful conditions may have to be introduced in order to assure additivity of the distances and consistency of the respective model. This author was unable to work out examples of more general and at the same time appropriate and fairly simple assumptions for the unknown function $p_0(t)$.

If $p = \text{const}$, then (7) under the restriction (4) furnishes an additive distance definition because:

$$\begin{aligned} -p[D_{ij} + D_{jk}] &= \log(1 - pc_{ij}) + \log(1 - pc_{jk}) \\ &= \log(1 - pc_{ij} - pc_{jk} + p^2 c_{ij} c_{jk}) = \log(1 - pc_{ik}) = -pD_{ik}, \end{aligned}$$

because of (4). Let us now investigate whether there is a consistent system of c.p.'s satisfying (4). Put, as in G.(48), $c_{i,i+1} = p_i$, combine (4) with G.(50) and write $p = 2\epsilon$. It follows that (4) is satisfied with $0 \leqq \epsilon \leqq 1$, if:

$$p_{ij} = \epsilon p_i p_j, \qquad p_{ijk} = \epsilon^2 p_i p_j p_k, \cdots. \tag{8}$$

Here p_{ij} is the probability of the simultaneous occurrence of the "events" numbered i and j, etc. For $\epsilon = 0$ we get "disjoint events" (see G.i) for the discussion of consistency). Assume now $\epsilon > 0$. By some considerations, analogous to those p. 54 G, the following necessary and sufficient condition of consistency follows:

$$\prod_{i=1}^{m-1} (1 - \epsilon p_i) \geqq 1 - \epsilon \qquad (\epsilon > 0). \tag{9}$$

This restriction (not considered by Haldane or Etherington) is, of course, relevant. If e.g. $m = 3$, $p_1 = p_2 = 4/5$, then ϵ must be $\geqq 15/16$; or if $m = 4$, $p_1 = p_2 = p_3 = \frac{1}{2}$, $\epsilon \geqq 3 - \sqrt{5}$ results. The restriction required by the "linear theory" is

$$(10) \qquad p_i \leqq \frac{1}{2\epsilon}, \qquad (i = 1, 2, \cdots, m - 1).$$

Hence this model is consistent under certain restrictions. It is, in contrast to Etherington's contention, different from iii) G. p. 54. The corresponding distance definition (7) is different from the author's. The D_{ij} thus defined are additive, and D_{ij} depends on c_{ij} only and not on the intermediate genes. The author's definition of distances, d_{ij}, is general, additive and seems to the author to be well adapted to the biological situation; since the definition of d_{ij} is not related to any particular model it is compatible with any model, which may contain any desired—consistent—assumptions about "interference," etc. For example in G. iv) p. 55, an n-parametric model has been suggested which seems fairly flexible.

It may however seem more acceptable to the biologist not to use a general distance definition but to define "distance" merely in relation to some sufficiently general "model" (such that the distance definition would vary with the model), instead of accepting an all-over definition as ventured in the author's paper. The particular model (8) in connection with its related distance definition (7) might give an example of such an approach.[3, 4]

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[3] As Etherington remarks, eq. (14′) in the author's original paper is not correct One can only state that (47) holds The mistake is however without consequence since no conclusions are drawn from (14′). The same mistake was pointed out by Professor Kai Lai Chung.

[4] Etherington writes: "I have been kindly allowed to read Professor Geiringer's MS. and feel that some comments are necessary

The standard procedure for calculating the distance between two linked genes is as follows. A selection of intermediate genes is taken and the adjacent crossover values calculated, giving a provisional estimate of the distance as in Geiringer's formula (1). When further intermediate genes are added to the selection, it is found that the provisional distance increases, but there is apparently a maximum value beyond which it cannot be increased. This unknown maximum value is the distance, and the geneticist accepts (1) as the distance when he is sure that he has observed a sufficient number of intermediate genes to give a good enough approximation to the true distance. Thus Geiringer's formula (1) gives the geneticist's true distance only on the understanding that it includes all genes intermediate between i and j; but generally speaking the great majority of these genes may be unobservable in the sense that they have no observably distinct alleles by means of which the c.p.'s could be calculated, though from time to time fresh genes may become observable by mutation.

In some cases the above procedure fails because not enough intermediate genes can be observed; then Haldane's analysis is useful It should be emphasized that his distance is additive by definition. (For a geometrical analogy, think of the genes as points closely distributed along a curve, chords representing c.p 's. Haldane's definition of the distance is analogous to defining arc length of the curve as a limiting sum of chords.) In my tran-

scription of his treatment, I should perhaps have made it clearer that the derived formula (6) gives only the distance D_{ij} measured from the initially chosen and fixed gene i to an arbitrary gene j. Other distances D_{jk}, $(i < j < k)$, are deduced from it by the postulate of additivity $(D_{jk} = D_{ik} - D_{ij})$. If the origin i is changed, there will be a similar formula (6), but it should not be assumed that the function p_0 is the same. In referring to certain conditions necessary 'to assure additivity,' Geiringer evidently means conditions that the function p_0 may be the same for all origins i. These conditions would be interpreted biologically as asserting uniformity of interference along the chromosome. I agree that there are further points to be cleared up in this connection.

If I might sum up the discussion, I would say that the geneticist's conception of the distance between genes is an actual property of the corresponding chromosome segment. Geiringer's definition represents the best possible general approach to this from the limited data of the l.d. alone. Haldane's definition fits the geneticist's conception, and his investigation is an attempt to get the best estimate of the distance by making approximate assumptions as to what happens between the observed genes. It is based on the unobservable crossover-distribution of a supposed infinite set of genes, but can be applied to particular models of this infinite c.d. so as to derive results which involve only a finite and observable c.d. Finally it should be mentioned that in the paper quoted, Haldane gave also an alternative method for the case $p = 2$, leading to the same formula (7'), which is really equivalent to defining the distance as the mathematical expectation of the number of chiasmata (not crossovers in G.'s sense) in the interval (i, j)."

A CRITERION OF CONVERGENCE FOR THE CLASSICAL ITERATIVE METHOD OF SOLVING LINEAR SIMULTANEOUS EQUATIONS

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The recent development of two devices[1, 2] for solving linear simultaneous equations by means of the classical iterative method[3] has stimulated the writer to investigate convergence criteria for the method. There are in the literature[4] necessary and sufficient criteria for convergence of symmetric systems, and sufficiency criteria for general systems. So far as the writer knows, however, this is the first development of a necessary and sufficient criterion for convergence in the general case. The results obtained are applicable to any arbitrary square non-singular matrix in which $a_{ii} \neq 0$.

Let the set of equations be represented by

$$AX = G, \tag{1}$$

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[1] Morgan, T. D., Crawford, F. W., "Time-saving computing instruments designed for spectroscopic analysis", *The Oil and Gas Journal*, August 26 (1944), pp. 100–105.

[2] Berry, C. E., Wilcox, D. E., Rock, S. M., Washburn, H. W., "A computer for solving linear simultaneous equations", to be published.

[3] Hotelling, Harold, "Some new methods in matrix calculation", *The Annals of Mathematical Statistics*, Vol. XIV (1943), pp. 1–34.

[4] Mises, R. von and Pollaczek-Geiringer, Hilda, "Zusammenfassende Berichte. Praktische Verfahren der Gleichungsauflosung" *Zeitschrift für angewandte Math. und Mechanik*, Vol. 9 (1929), pp. 58–77, and 152–161.

in which A is the square matrix of the coefficients, X is the column matrix of the unknowns, and G is the column matrix of the constant terms. $|A|$ is the determinant of A.

We define a matrix A_1 which contains the prediagonal and diagonal terms of A, and a matrix A_2 which contains the postdiagonal terms of A. According to this definition,

$$A_1 + A_2 = A. \tag{2}$$

In the classical iterative method, arbitrary (or approximate) values of the x's are chosen, the first equation is solved for the first unknown, the second equation for the second unknown, etc , using in each equation the most recent approximations to the x's. This process may be written

$$A_1X^{(1)} + A_2X^{(0)} = G, \tag{3}$$

in which $X^{(0)}$ is the initial approximation matrix, and $X^{(1)}$ is the approximation matrix existing at the end of the first iterative cycle The superscripts indicate the number of the approximation. The next cycle is described by

$$A_1X^{(2)} + A_2X^{(1)} = G, \tag{4}$$

and the mth by

$$A_1X^{(m)} + A_2X^{(m-1)} = G. \tag{5}$$

The method yields a solution, i.e., converges, if

$$\lim_{m\to\infty} (X^{(m)} - X) = 0.$$

Solving (5) explicitly for $X^{(m)}$,

$$X^{(m)} = A_1^{-1}G - A_1^{-1}A_2X^{(m-1)}. \tag{6}$$

Subtracting X from each side,

$$X^{(m)} - X = A_1^{-1}G - A_1^{-1}A_2X^{(m-1)} - X, \tag{7}$$

and making use of (1) and (2)

$$X^{(m)} - X = -A_1^{-1}A_2(X^{(m-1)} - X) \tag{8}$$

Since (8) applies for any value of m, we may write

$$X^{(m)} - X = (-A_1^{-1}A_2)^2(X^{(m-2)} - X), \tag{9}$$

and continuing this process,

$$X^{(m)} - X = (-A_1^{-1}A_2)^m(X^{(0)} - X). \tag{10}$$

Now, $\lim_{m\to\infty} (X^{(m)} - X) = 0$ if and only if

$$\lim_{m\to\infty} (-A_1^{-1}A_2)^m = 0. \tag{11}$$

This is a general result, applicable to any arrangement of the terms of an arbitrary square matrix A, subject only to the conditions that $| A | \neq 0$ and that no diagonal term of A is zero. In this latter exceptional case, the iterative method itself obviously cannot be applied.

The criterion (11) clearly shows that the order in which the elements of the matrix A are arranged is important. For instance, it is plain that an arrangement in which the diagonal terms are large and the off-diagonal terms, particularly the post-diagonal terms, are small will tend to favor convergence.

A somewhat relaxed condition, which is sufficient but not necessary, is obtained through the use of an inequality used by Hotelling[3], namely,

$$N(B^m) \leq [N(B)]^m, \tag{12}$$

in which $N(B)$ is the norm of the matrix B, that is, the square root of the sum of the products of its elements by their complex conjugates, or in the case of a real matrix the square root of the sum of the squares of the elements.

The condition is that, if

$$N(A_1^{-1}A_2) < 1, \tag{13}$$

then

$$\lim_{m \to \infty} (A_1^{-1}A_2)^m = 0. \tag{14}$$

Criterion (13) is readily computed, since A_1^{-1}, the reciprocal of a triangular matrix is readily computed, and the post-multiplication by A_2 involves a number of zero terms.

A more stringent condition than (13) though still not a necessary condition, is that if some finite number p can be found such that

$$N(A_1^{-1}A_2)^p < 1, \tag{15}$$

then (14) follows. Since n matrix squarings result in a value of $p = 2^n$, the size of the norm for fairly large values of p can be investigated without excessive labor.

A REMARK ON INDEPENDENCE OF LINEAR AND QUADRATIC FORMS INVOLVING INDEPENDENT GAUSSIAN VARIABLES

By M. Kac

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The purpose of this note is to call attention to the following useful theorem, which to the best of my knowledge was never stated explicitly.

If $X_1, X_2, X_3, \cdots X_n$ are identically distributed, independent Gaussian random variables each having mean 0, *then the necessary and sufficient condition that*

$$\sum_{j,k=1}^{n} a_{jk} X_j X_k \quad \textit{and} \quad \sum_{j=1}^{n} \alpha_j X_j = \alpha \cdot X$$

be independent, is that

$$A\alpha = 0,$$

where A is the matrix of the quadratic form, α the vector $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ and X the vector $(X_1, X_2, \cdots X_n)$.

PROOF OF SUFFICIENCY.[1] Since $A\alpha = 0$, it follows that 0 is an eigenvalue of A, and α is a corresponding eigenvector.

Denoting by $\lambda_2, \cdots, \lambda_n$ the remaining eigenvalues and by $\beta_2, \cdots, \beta_n$ the corresponding eigenvectors, we have

$$\sum_{j,k=1}^{n} a_{jk} X_j X_k = \sum_{j=2}^{n} \lambda_j (\beta_j \cdot X)^2.$$

Since the β's are orthogonal to α, it follows that the linear combinations $\beta_j \cdot X$ are independent of $\alpha \cdot X$, and this completes the proof.

PROOF OF NECESSITY. From the assumption of independence it follows that

$$\sum_{j,k=1}^{n} a_{jk} X_j X_k \quad and \quad \left(\sum_{j=1}^{n} \alpha_j X_j\right)^2 = \sum_{j,k=1}^{n} \alpha_j \alpha_k X_j X_k$$

are independent. Thus by Craig's theorem[2]

$$AB = 0$$

where $B = ((\alpha_j \alpha_k))$.

This implies almost immediately that $A\alpha = 0$.

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[1] *Added in proof.* Dr. L. Guttman has kindly pointed out to me that the proof of sufficiency given here has been used by D Jackson in the article "Mathematical principles in the theory of small samples", *Amer Math. Month*, Vol 42 (1935), pp 344–364, see in particular pp. 354–355 Jackson considers only the independence of $\bar{x}$ and s^2, which is of crucial importance in deriving student's distribution.

[2] A. T. CRAIG, *Annals of Math. Stat.*, Vol. 14 (1943), pp. 195–197, see also H. HOTELLING, ibid., Vol. 15 (1944), pp 427–429

ABSTRACTS OF PAPERS

Presented on September 16, 1945 at the Rutgers meeting of the Institute

1. On The Variance of a Random Set in n Dimensions. HERBERT ROBBINS, Lieutenant USNR Postgraduate School, Annapolis, Md.

Using a general formula for the moments of the measure of a random set X (*Ann. Math. Stat.* Vol. XV (1944), pp. 70–74) we find the mean and variance in the case where X is a random sum of n-dimensional intervals with sides parallel to the coordinate axes, thus generalizing the results previously found (loc. cit.) for the case $n = 1$.

2. The Non-Central Wishart Distribution and its Application to Problems in Multivariate Statistics. T. W. ANDERSON, Princeton University.

The non-central Wishart distribution is the joint distribution of sums of squares and cross-products of deviations of observations from multivariate normal distributions with identical variance-covariance matrices and with different sets of means. The rank of the non-central Wishart distribution is defined as the rank of the matrix of sets of means. In a previous paper (by M. A. Girschick and the present author) the non-central Wishart distribution is given explicitly for the rank one and two cases and indicated for the case of any rank. In the present paper the characteristic function of the non central Wishart distribution is given for general rank. The distribution, which is given in the form of a multiple integral, is the product of a central Wishart distribution and a symmetric function of the roots of a determinantal equation involving the matrix of squares and cross products of observations and the matrix of population means. It is shown that the convolution of two non-central Wishart distributions is again a non-central Wishart distribution if the variance-covariance matrices are the same. The moments of the generalized variance and the moments of the likelihood ratio criterion for testing certain linear hypotheses (for example, the hypothesis that the means of a set of populations are identical, given that the matrices of population variances and covariances are the same) are obtained for the linear and planar non-central cases in terms of infinite series. Likelihood ratio criteria are developed for testing the dimensionality of the means of a set of multivariate populations (with identical variances and covariances) on the basis of one sample from each. The criterion for testing whether the dimensionality is h in the space of p dimensions is a symmetric function of $p - h$ smallest roots of the determinantal equation involving the sample estimate of the matrix of variances and covariances and the sums of squares and cross-products of deviations of sample means. The maximum likelihood estimate of the hyperplanes and positions of means on them are obtained. The asymptotic distributions of the criteria are χ^2-distributions.

3. The Effect on a Distribution Function of Small Changes in the Population Function. BURTON H. CAMP, Wesleyan University.

It is generally assumed in the application of distribution theory that, if the actual population function is not very different from the one used in the theory, then the true sampling distribution of a statistic will not be very different from the one obtained in the theory. But elsewhere in mathematics we do not assert that a conclusion will be only slightly modified by a small deviation in the hypothesis. This paper presents some theorems which are useful in determining the maximum effect on a sampling distribution of certain kinds of small changes in the population function.

4. Composite Distributions. CASPER GOFFMAN and BENJAMIN EPSTEIN, Westinghouse Electric Corporation.

Let $f(x, \theta_1, \theta_2, \cdots, \theta_n)$ be a function such that for every point $\theta_1 = \theta_{10}, \cdots, \theta_n = \theta_{n0}$ in parameter space, x is a random variable with p d f. $f(x; \theta_{10}, \cdots, \theta_{n0})$. Suppose further that the parameters $\theta_1, \theta_2, \cdots, \theta_n$ are themselves random variables whose p d.f 's are given respectively by $\phi(\theta_1), \cdots \phi(\theta_n)$ Using a concept of "probability contained in an interval" and an axiom based on this concept, we show that x is a random variable with p d.f. $g(x)$ given by the formula

$$(1) \qquad g(x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x, \theta_1 \cdots, \theta_n) \phi(\theta_1) \cdots \phi(\theta_n)\, d\theta_1 \cdots d\theta_n .$$

In this paper we consider statistical properties of the function $g(x)$ in cases of particular interest in applications. The cases treated here are (a) where the mean, $\bar{x}$, is the only variable parameter, (b) where the standard deviation, σ, is the only variable parameter, and (c) where the mean $\bar{x}$, and the standard deviation, σ, are both variable parameters, $\bar{x}$ and σ being independent

It is shown that problems (a) and (b) are equivalent respectively to the sum and product of two independent random variables, one of which has zero mean. Formulae for the moments in problem (c) are then derived in terms of the formulae obtained for (a) and (b).

5. Population, Expected Values and Sample. E. J. GUMBEL, New School for Social Research.

Let x be an unlimited continuous variate, and let $F(x)$ be the probability of a value equal to, or less than, x. Then the expected m^{th} values $\bar{x}_m$, for n observations, are approximations to the most probable m^{th} values and defined by $F(\bar{x}_m) = F_1 + (F_n - F_1)(m - 1)/(n - 1)$, where F_1 and F_n are the probabilities of the most probable first and the most probable last value. The probabilities F_1, $1 - F_n$ and $(F_n - F_1)/(n - 1)$ are of the order of magnitude $1/n$.

The distribution of the expected values $\bar{x}_m$ differs from the distribution of the sample and from the theoretical distribution. However, for a symmetrical distribution the mean and the odd moments about mean calculated from the expected values coincide with the mean and the moments of the population. For the normal distribution, the expected standard deviation $\sigma(n)$ divided by the standard deviation σ of the population and traced on normal probability paper approximates a linear function of $\sqrt{\log n}$. The approach of $\sigma(n)$ toward σ is slow For 500 observations, $\sigma(n)$ is about 99% of σ The moments of the distribution of the expected values exist even in the case that the moments of the theoretical distribution diverge.

6 On Optimum Estimates for Stratified Samples. MORRIS H. HANSEN and WILLIAM N. HURWITZ, Bureau of the Census.

A stratified sample is drawn from a population with R strata. Neyman found the optimum sample allocation for the "best unbiased linear estimate." However, biased but consistent estimates of the form $\frac{x_i'}{y_i'}$ where both x_i' and y_i' are random variables have been found to give more reliable results in a large class of problems. Even more efficient estimates can be obtained by finding the values of n_i (the sample size) and w_i which minimize the mean square error of estimates of the form $\Sigma w_i \frac{x_i'}{y_i'}$ or $\frac{\Sigma w_i x_i'}{\Sigma w_i y_i'}$.

7. Pearsonian Correlation Coefficients Associated with Least Squares Theory.
PAUL S. DWYER, University of Michigan. (Read by Title).

In least squares theory we have the predicting variable x, the observed value of the predicted variable, y, the residual e, and the predicted value of the predicted variable y. The purpose of this paper is to study the Pearsonian coefficients resulting from correlating all these variables in pairs (a) in the case of a single predicted variable and (b) in the case of two or more predicted variables. The results yield such coefficients as multiple correlation, multiple alienation, partial correlation, part correlation, and new coefficients not previously in use. The results are given in expanded, determinant, and matrix form. A simplified calculational technique is provided.